Strategic Interactions in Financial Networks

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Abstract

This paper models interactions of firms in a pre-trading(fixed network of lending/borrowing) period whereby firms set fixed lending rates given loan management cost. We show strategic substitution in the rate each firm sets and more fundamentally, propose that the rates charged to debtors by a creditor firm is likened to results from a private provision of public good in networks game. We then highlight specific core-periphery network properties in relation to interdependence and Nash rate charged by firms. For welfare policies, we find neutrality of intervention policies that create or reduce transaction cost and improvement based on policies that provide administrative subsidies thus creating an avenue for cost effective resource transfer policy. Lastly, we find significant relationship between a firms centrality measured by weaker negative externality and welfare improvement due to such subsidy.

JEL classification: C72, D44, D61, D62, D85, E43, H23, H41.

Keywords— Externalities, Nash-Equilibrium, Centrality

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1 Introduction

Financial behaviours can be studied in several ways including interactions which are a function of the link between parties(firms) involved. Links commonly modeled are those which captures contractual obligation of one party which represents an asset to the other party. Because a contract can only be created by the mutual consent of the two potential firms involved, links project such attributes among a host of properties. Also with these links comes exposure. Sometimes these exposure could be advantageous to firms for example in a firms ability to external shocks and promotion of stability (Elliott, Golub, & Jackson, 2014). However, where these links are binding and difficult to sever, they could be a source of negative spillover to one or more firms.

We explore endogenous interest(lending) rate determination in fixed lending/borrowing network where firms account for administrative overheads associated with levels of total lending and a percentage of borrowing. Inter-firm lending rate, as core of the contract clause, is the additional amount over a period of time to which a lending firm charges a borrowing firm for such transaction. In this case, firms give significant attention to cost arising from its debt management procedure. It is not a novel observation that the vast amount of businesses engage in credit financing. Many of such businesses have both creditors (suppliers, banks etc) as well as debtors (customers,retailers, etc). In order to effectively keep track and ensure a smooth settling process, administrative resources are then devoted towards different aspects of such overall loans. One major assumption we make is to relate the total amount of loans[1] to total administrative cost. The intuition here remains that large amount of loans implies large volume of transaction to which raises risk of significant amount of loss to both the personnel[2] involved and the business at large. Such delicate nature then spurs the need for remuneration to match up to such risk, hence greater expenses. Other types of market frictions could also add to this cost. This is so that assuming a single clearing period, all liabilities are settled such that the lending firm regains its cash and some premium.

As a main result, we show that the optimal lending rate a firm charges to its debtors is on derived from a game of strategic substitution. such substitution behaviours are in line

[1] We discount those from creditors as we show later in the model.

[2] Accountant, Debt Administrator, Legal teams, etc.

with major public goods in network literature such as Bramoullé, Kranton, and D'amours (2014), Allouch (2015), etc. We then show that Nash equilibrium exists and is uniquely defined in pure strategies given our network game. We then discuss special equilibrium properties in networks such as the Core-periphery network which are specifically well bounded. We find that core firms are closely dependent on links from other core firms while giving less priority to periphery firms. Periphery firms on the other hand substitute from rates charged by their core interlinks.

Adopting a utilitarian welfare approach, we show welfare neutrality of firms, whose equilibrium lending rate is positive, to intermediation policies. Given such neutrality, we hold that Pareto improvement of welfare can be achieved at zero cost to planner by leaking out little amount from the paying system (such that active firms are fixed) and proportionally splitting it so as to subside for loan management expenses. Lastly, we then show quality of firms least externality based centrality is vital for targeting firms with subsidy policies.

1.0.1 Related Literature

Rates agreed upon by such parties become an asset in that they contribute potentially to profit of firms from an inter-firm lending in a given trading period. Sometime these rates are determined based on set regulatory benchmarks, for example in case of banks premiums are added to the inter-banks offered rate (e.g Libor as shown by Eisl, Jankowitsch, and Subrahmanyam (2017), Coulter, Shapiro, and Zimmerman (2017), Duffie and Dworczak (2014), Abrantes-Metz, Kraten, Metz, and Seow (2012) and Eaglesham (2013)). There might be other benchmark at typical firms level which for example might be the risk-free interest rate used in CAPM analysis. We however pay less attention to such parameter. Instead we focus on situations where by firms primarily make decisions as to their lending rate. An existing work in this is found in Aldasoro, Gatti, and Faia (2017) which models an endogenous interest premium in a system in which firms given potential cascading defaults/systemic risk. Hence, the additional premium is determined while accounting for estimated default probability.

Earlier discussion has made reference to strategic interactions and more sepcifically, a substitution relationship. Strategic interdependence here can be traced to network properties of our debtors and/or creditor firms as a form of financial network. Most financial network literature such as Morris (2000), Morris and Shin (2001), Allen and Babus (2008), Babus (2016), Bhattacharya, Gale, Barnett, and Singleton (1985) have focused on systemic risk as well as other issues to do with risk contagion and financial network stability e.g Caballero and Simsek (2013), Cabrales, Gottardi, and Vega-Redondo (2014), Nier, Yang, Yorulmazer, and Alentorn (2007), Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), Greenwood, Landier, and Thesmar (2015)(on banks), Galeotti, Ghiglinoy, and Goyal (2016), König, Tessone, and Zenou (2009), Bilkic, Gries, et al. (2014), Gollier, Koehl, and Rochet (1997), etc. Others on network influence and power as in Demange (2016) as well as Aldasoro and Angeloni (2015) while a host of literature pay attention to liquidation as well as network financing such as Allouch and Jalloul (2016), Amini, Filipović, and Minca (2016), Feinstein (2017), Rogers and Veraart (2013) and Elsinger, Lehar, and Summer (2006) to mention but a few. However, it is noted that since systemic risk and stability is the key focus of most of the works mentioned above, strategic interaction plays less importance.

Additionally, little attempt have been given to link interactions in debt networks to public good games which yields best replies revealing strategic substitution. However, those on contagion in the previous par graph rely on strategic complements. Public good games with strategic substitution are found in in Allouch (2015) and Allouch and King (2018a)(which shows equilibrium in a fully bounded action profile). Also importantly is Bramoullé and Kranton (2007) and Bramoullé et al. (2014) where by interaction mostly based on strategic substitution is identified. Such games of public good provision and more specifically private public good provision can materialize in different ways in financial networks. That being said, the underlying ideas has not been aimed at identifying such behaviors in financial network games. A contrast of this work from seminal works including Bramoullé and Kranton (2007) and Allouch (2015) is the close attention paid to interactions to undirected network (with Bramoullé et al. (2014) providing initial intuitions as to weighted and direct network). While there are observable differences, intuitions are very useful in observing such behaviors in financial networks which are uni-directional and weighted in nature.

Our results on neutrality is also in distinction to neutrality of income redistribution.

For income redistribution, neutrality holds such that wealth transfer between active agents in a public good game leads to no change in aggregate public good provision and individual consumption. These forms of neutrality is discussed well in Bergstrom, Blume, and Varian (1986), Wells (2004) as well as Allouch (2015) though we point out that our intervention are not particularly redistributive in nature. lastly, our targeting criterion has a lot of similarities to works like key player concepts in works like Ballester, Calvó-Armengol, and Zenou (2006), Galeotti, Golub, and Goyal (2020), Belhaj, Bervoets, and Deroïan (2016) as well as Belhaj and Deroïan (2019).

2 The Model

Assume a three period economy consisting of $\hat{\mathcal{N}} = \{1, \ldots, n\}$ set of firms. We denote the set of periods denoted as $\mathcal{T}$ such that $\mathcal{T} = \{t-1, t, t+1\}$. For every firm $i \in \hat{\mathcal{N}}$, its neighborhood is denoted as $\mathcal{N}$ and $\mathcal{N}_i = \{\mathcal{N}_i^{out} \cup \mathcal{N}_i^{in}\} \subset \hat{\mathcal{N}}$ where $\mathcal{N}_i^{out}$ represents firm $i \in \mathcal{N}$'s debtors and $\mathcal{N}_i^{in}$ represents firm $i \in \mathcal{N}$'s creditors. Debtors are those whom the firm lends to and creditors are who the firm lends from. The amount to be borrowed by each firm $i \in \hat{\mathcal{N}}$ is given as $b_i : b_i > 0 \quad \forall\ i \in \hat{\mathcal{N}}$. This interaction forms a borrowing network $\mathbf{g}(\hat{\mathcal{N}}, \hat{g})$ with g representing links between firms.

Each firm strictly lends to each other based on the network $\mathbf{g}(\hat{\mathcal{N}}, \hat{g})$ in t and as such $\mathbf{g}(\hat{\mathcal{N}}, \hat{g})$ indicate borrowing/lending contract in this model. The network $\mathbf{g}(\hat{\mathcal{N}}, \hat{g})$ is set at $t-1 \in day$ so that it is exogenous to t onward. At $t+1$ links are dissolved (cleared). Given that if firm $j \in \mathcal{N}_i^{in}$, then $\hat{g}_{ji} > 0$ while being zero otherwise, then $\sum_{j \in \mathcal{N}_i^{in}} \hat{g}_{ji} = 1$ and $\sum_{j \in \mathcal{N}_i^{in}} \hat{g}_{ji} b_i = b_i$. A firm while lending charges an extra amount it sets at $t \in day$ denoted as r_i. This rate r_i remains fixed for the rest of $\mathcal{T}$ once set. Also, we assume that firms then incur additional cost to manage debtors and creditors accounts and repayment procedures which we capture under loan management cost.

In the economy, a typical firm i's lending is then given as as;

$$b_{-i} = \sum_{j \in \mathcal{N}_i^{out}} \hat{g}_{ij} b_j, \qquad \forall\ i \in \hat{\mathcal{N}}. \tag{2.1}$$

As such its balance sheet at $t+1$ is given as follows;

Table 1: Firms balance Sheet at $t + 1$

Assets	Amount	Liability	Amount
Debtors	$xxxx$	Creditors	$xxxx$
		Profit	$xxxx$

So then assume a firm $i \in \hat{\mathcal{N}}$ who is scheduled at t to lend to as well as borrow from other firms within the same system (Thus creating incoming and/or outgoing links). Then profit for the firm $i \in \hat{\mathcal{N}}$ intuitively drawn from its balance sheet marks its payoff which we show subsequently.

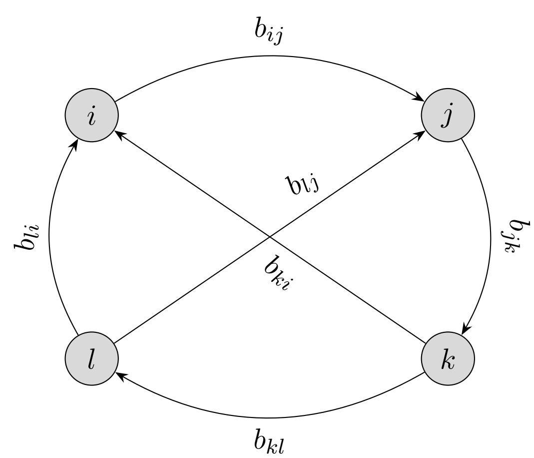
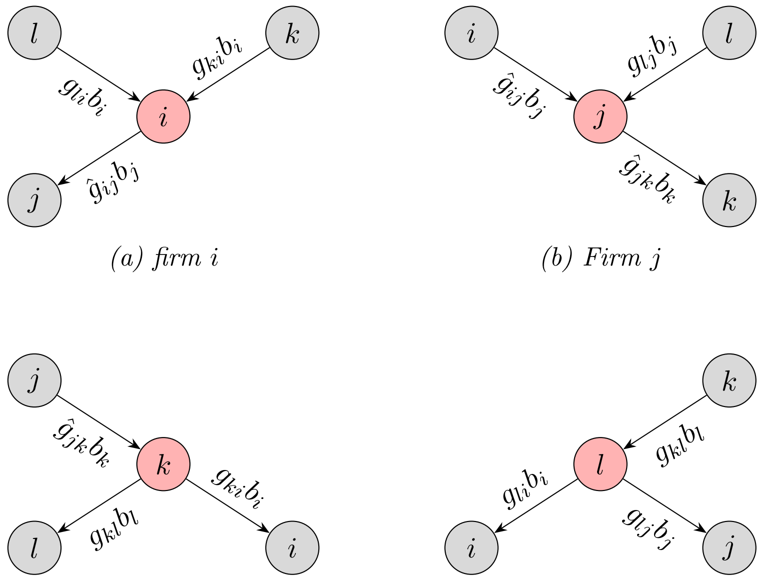


Figure 1: A Bounded Borrowing Network:
Arrows (edges) point to direction of borrowers and originate from lenders.

We observe that the firm $i \in \hat{\mathcal{N}}$ is only concerned about his lenders and borrowers as opposed to the entire network. Since we ignore default risk it then implies that regardless of the nature of borrowing network, each firm i can identify its position given a star which carries its lenders and borrowers.

Assume for now that $\hat{\mathcal{N}} = \{i, j, k, l\}$. Observe from $fig.$ 1 and the break down in $fig.$ 2. It shows that from any directed network of borrowing and lending, for example the network in $fig.$ 1, it can further be broken down into sub-networks as in $fig.$ 2 thus capturing each individual firms lending and borrowing.



(a) firm i *(b) Firm j*

(c) Firm k *(d) Firm l*

Figure 2: Decomposed Network to capture pivotal links. We see that for decision making purpose, it is direct incoming and outgoing link that are useful to firms decision. This is as Loan management cost is spent mainly through managing a firms asset/liabilities.

To capture this into the firms' payoff in the simplest, assume the firm $i \in \hat{\mathcal{N}}$ has included in its cost, the cost of loan management. This loan management cost is split into 2 main parts. The first a homogeneous constant κ which measures the level of efficiency in managing overall debtors and creditors accounts and recovery process. In itself, a higher κ implies lesser efficiency in loan management while a lesser κ implies greater efficiency. Such efficiency could arise from specialization, technical know-how, technological progress and other factors that imply positive economies of scale for the firm. The second part is the endogenous loan size parameter which we denote as μ_i for the given firm $i \in \hat{\mathcal{N}}$. More formally, we define $\mu_{i\in\hat{\mathcal{N}}}$ as follows;

$$\mu_i(r_i, r_{-i\in\mathcal{N}_i^{in}}) = b_{-i} \bullet r_i + a \sum_{j\in\mathcal{N}_i^{in}} r_j \left(\hat{g}_{ji} b_i\right), \tag{2.2}$$

where the parameter $a \in \mathbb{R}_+$ captures the degree to which interest from debtors increases administrative cost. Properties of this parameters are crucial to establish uniqueness which is captured subsequently. Let $\mathbf{r} = (r_i)_{i\in\mathcal{N}} \in \mathbb{R}_+^n$ be the lending rate vector for firms, we assume that the firm $i \in \hat{\mathcal{N}}$ has the following variable loan management cost;

$$\kappa \bullet f(\mu_i(r_i)) \tag{2.3}$$

such that we then have the following important assumption;

Assumption 1. *$\forall$ firm $i \in \hat{\mathcal{N}}$, we hold that;*

$$\frac{\partial f(\mu_i(r_i))}{\partial r_i} > 0, \quad \frac{\partial^2 f(\mu_i(r_i))}{\partial r_i^2} > 0.$$

We assume convexity here due to the fact that we assume that cost exponentially rises as total variable loan obligation to and from the firm rises. The (2.3) captures the variable loan management cost which is weighed using the parameter κ that would usually assume a very small vale. Loan management cost as defined here includes both the firm i's debtors management as well as its creditors. We have provided the justification behind this in earlier sections.

2.1 Strategic Substitution

We then hold that the firm $i \in \hat{\mathcal{N}}$, $f(\mu_i) = (\mu_i)^2$ which fits well into assumption 1. Then the firm $i \in \hat{\mathcal{N}}$, has the following payoff function,

$$P_i(r_i|r_j, \ldots) = b_{-i} \bullet r_i - \sum_{j \in \mathcal{N}_i^{in}} r_j \left(\hat{g}_{ji} b_i\right) - \kappa(\mu_i)^2. \tag{2.4}$$

To elaborate, the payoff captures variable components of a firm $i \in \hat{\mathcal{N}}$ since for example, lends out a total of b_{-i} and at $t+1$, gets back $b_{-i} \bullet (1 + r_i)$ from its debtors. So what it gets at $t+1$ is $\overline{b_{-i}} + b_{-i} r_i$. Additionally, it pays $\sum_{j \in \mathcal{N}_i^{in}} (1 + r_j) \left(\hat{g}_{ji} b_i\right)$ to its creditors such that it is broken into $\overline{\sum_{j \in \mathcal{N}_i^{in}} \left(\hat{g}_{ji} b_i\right)} + \sum_{j \in \mathcal{N}_i^{in}} r_j \left(\hat{g}_{ji} b_i\right)$. We thus define the firm $i \in \hat{\mathcal{N}}$ payoff as one that captures only the parts which are multiples of the action profile $\mathbf{r}$.

To optimize P_i, the Lagrange equation is given as;

$$\max_{r_i} Q_i(r_i) = b_{-i} \bullet r_i - \sum_{j \in \mathcal{N}_i^{in}} \hat{g}_{ji} b_i \left(r_j\right) - \kappa(\mu_i)^2 - \Psi r_i \tag{2.5}$$

With the complementary slackness condition $\Psi r_i = 0$ being such that $\Psi = 0$, the *FOC* thus equates the marginal benefit to marginal cost ($MB_i = MC_i$) which is given as;

$$\frac{\partial Q_i}{\partial r_i} = 0 \Rightarrow 2b_{-i}(\mu_i) = \frac{b_{-i}}{\kappa}.$$

We then make r_i the subject of the formula using μ_i as in (2.2) so that we have the firm i's optimal lending rate as,

$$r_i = \frac{1}{2\kappa b_{-i}} - a \sum_{j \in \mathcal{N}_i^{in}} \frac{\hat{g}_{ji} b_i}{b_{-i}} r_j. \tag{2.6}$$

Let us have $\pi_i = \frac{1}{2\kappa b_{-i}}$ and $g_{ji} = \frac{\hat{g}_{ji} b_i}{b_{-i}} = \frac{b_{ji}}{b_{-i}}$ $\forall$ $j \in \mathcal{N}_i^{in}$, we have;

$$r_i = \pi_i - a \sum_{j \in \mathcal{N}^{in}} g_{ji} r_j. \tag{2.7}$$

The linear reaction curve (best reply) for the $firm - i$ when $r_{i \in \hat{\mathcal{N}}} \in [0, \mathbb{R}^+]$ is given as;

$$r_i = \max \left\{ \pi_i - a \sum_{j \in \mathcal{N}^{in}} g_{ji} r_j, 0 \right\}. \tag{2.8}$$

The lending rate π_i reflects the autarkic amount charged to each firm j such that $j \in \mathcal{N}_i^{in}$. firm i is desires a greater r_i if it expects to lend in greater deal compared to its borrowing and thus submits its rate accordingly. However, the magnitude of its rate charged depends on its best reply. π_i additionally reveals the Engels curve for the firm i. Also, strategic substitution properties is captured in $\frac{\delta r_i}{\delta r_j} = -a g_{ji}$ for $j \in \mathcal{N}_i^{in}$.

Let $\mathbf{G} = [g_{ji}]$ be a zero-diagonal matrix and the game arising from (2.8) be denoted as $\Gamma(\mathbf{G}, \mathbf{r}, a)$. We make distinction between participating firms and those who do not participate in $\Gamma(\mathbf{G}, \mathbf{r}, a)$. This is because financial networks could posses cyclical interconnection as we see in line works within Eisenberg and Noe (2001) framework. Assume a subset $\mathcal{S} \subset \hat{\mathcal{N}}$. We have the formal definition;

Definition 1. *A firm $i \in \hat{\mathcal{N}}$ is a sink-node* $\iff$ $\mathcal{N}_i^{out} = \{\}$.

Let also $\mathcal{N} \in \hat{\mathcal{N}}$ so that $|\mathcal{S} \cup \mathcal{N} = \hat{\mathcal{N}}$ meaning that a firm $i\mathcal{N}$ lends to at least one other firm. A Sink node is a debtor to one or more firms but does not lend out. This distinction is important for example if we have a firm i such that $\mathcal{N}_i^{out} = \{\}$, then $g_{ji} = \infty$ as $b_{-i} = 0$. It means are unable to define firm i's best reply as it makes no decision. Furthermore, we could have also the firm i such that $\mathcal{N}_i^{in} = \{\}$. Let $\mathbf{G}_i$ represent the $i-th$ row of the matrix $\mathbf{G}$, we would have $\mathbf{G}_i = (0)_{i \in \mathcal{N}}$ leading to a pure strategy Nash equilibrium $r_i = \pi_i$. This is described as *strategic dominance* as its lending rate is made in isolation. To avoid these instances, we introduce another important but common concept to directed networks as follows;

Definition 2. *A directed graph* $\mathbf{g}(\mathcal{N}, g)$ *is* strongly connected *(SC) if and only if for every* $\{0, n\} \in \mathcal{N}$*, there exist a* ***closed directed walk*** *(the sequence* $0, g_{01}, 1, g_{12}, \ldots, g_{n-1,n}, n, g_{0,n}, 0)$ *from* 0 *to* 0*.*

Then going further, we will rely on the assumption written below;

Assumption 2. *The graph* $\mathbf{g}(\mathcal{N}, g)$ *is strongly connected so that the set* $\forall$ *firm* $i \in \mathcal{N}$*, firm* i *is a strongly connected firm(SCF).*

This as such ensures that we avoid dominant equilibrium outcomes or undefined best replies given sink nodes (for any firm $i \in \mathcal{S}$, $b_{-i} = 0$ such that $r_i = \infty$).

3 Pure Strategy Solutions

We define in this section the shape and characteristics of equilibrium under such game $\Gamma(\mathbf{G}, \mathbf{r}, a)$.

3.1 Uniqueness and Stability

We present the existence of the equilibrium and conditions for uniqueness. To support our next few results, we define a key attribute which is positive definiteness. Given the vast amount of network literature emphasising on symmetric matrix, we define positive definiteness given our model as follows;

Definition 3. *Let* $\mathbf{M}$ *be a matrix and* $\nu_1(M), \ldots, \nu_n(M)$ *be the eigenvalues of the matrix*

$(\mathbf{M})$. *Then M is positive definite if and only if it holds that;*

$$\nu_1\left(\frac{\mathbf{M}+\mathbf{M}^T}{2}\right),\ldots,\nu_n\left(\frac{\mathbf{M}+\mathbf{M}^T}{2}\right) > 0. \tag{3.1}$$

Let the minimum eigenvalue of a matrix $\mathbf{M}$ be denoted as $\nu_{min}(\mathbf{M})$, we have the following lemma;

Lemma 1. *The matrix $(\mathbf{I}+a\mathbf{G})$ is positive definite in so far $a \in \left]0, \frac{1}{\left|\nu_{min}(\frac{\mathbf{G}+\mathbf{G}^T}{2})\right|}\right[$.*

Proof. See Appendix for proof. ■

For asymptotic properties of Nash equilibrium for firms lending rate interrelationship, we identify stability as the ability for small changes in rates charged in a certain period that differs from that suggested by the best reply converging back to the steady Nash equilibrium over when the game is repeated over and over. We summarize these properties under the following proposition;

Proposition 1. *Given the parameter $'a'$ meets the boundary conditions as in lemma 1, there always exists a unique Nash equilibrium in pure strategies for the game $\Gamma(\mathbf{G},\mathbf{r},a)$ and the unique Nash equilibrium in pure strategies of the game $\Gamma(\mathbf{G},\mathbf{r},a)$ is always asymptotically stable.*[3]

Proof. From Rosen (1965) concept of *diagonal strict concavity*, we understand that a sufficient condition for the payoff $P(\mathbf{r})$ to be diagonally strictly concave, then $\mathbf{H}(\mathbf{r},\mathbf{1})+\mathbf{H}(\mathbf{r},\mathbf{1})^T$ must be negative definite where $\mathbf{H}(\mathbf{r},\mathbf{1})$ is the Jacobian with respect to $\mathbf{r}$ of $P^{'}(\mathbf{r})$. Since it hold that the Jacobian $\mathbf{H}(\mathbf{r},\mathbf{1}) = -(\mathbf{I}+a\mathbf{G})$, then the condition is achieve should $(\mathbf{I}+a\mathbf{G})$ be positive definite which lemma 1 satisfies. It is then shown that Nash equilibrium is unique if and only if lemma 1 is satisfied. ■

This is so that each firm capture the amount charged by their creditors on borrowings in order to determine the rate charged to debtors without consideration for their own power.[4]

[3] This follows uniqueness results from Rosen (1965) and also Bramoullé et al. (2014) where it shows that unique Nash equilibrium is always asymptotically stable.

[4] The magnitude to which a firm $i \in \mathcal{N}$ rate charged r_i affects all other firms outcome.

3.2 Analysis of Equilibrium

Assume $\boldsymbol{\pi} = (\pi_i)_{i\in\mathcal{N}} \in \mathbb{R}^n_+$ is also a column vector while $\mathbf{r}^* = (r^*_i)_{i\in\mathcal{N}} \in \mathbb{R}^n_+$ is the Nash equilibrium vector. Following the best reply in (2.8), draw distinction between active and inactive firms in the definition below.

Definition 4. *A firm $i \in \mathcal{N}$ is thus defined as* active *if and only if $r^*_i(\mathcal{N}, a) \in]0, \mathbb{R}_+]$ and non-active if $r^*_i(\mathcal{N}, a) = 0$.*

Let the set of active firms be denoted with the set $\mathcal{A} \subseteq \mathcal{N}$ and hence non-active firms be $\mathcal{N} - \mathcal{A} \subsetneq \mathcal{N}$. Then using intuitions from Bergstrom et al. (1986), Bramoullé et al. (2014) and more closely,Allouch (2015), we have the following;

Proposition 2. *A set of rates vector $\mathbf{r}^*(\mathcal{A}, a)$ with active firms $\mathcal{A} \neq \{\}$ is a Nash equilibrium $\iff$ the following conditions hold true;*

1.

$$(\mathbf{I} + a\mathbf{G})_{\mathcal{A}\times\mathcal{A}} \bullet \mathbf{r}^*_{\mathcal{A}} = \boldsymbol{\pi}_{\mathcal{A}}$$

2.

$$a\mathbf{G}_{\mathcal{N}-\mathcal{A}\times\mathcal{A}} \bullet \mathbf{r}^*_{\mathcal{A}} \geq \boldsymbol{\pi}_{\mathcal{N}-\mathcal{A}}$$

Proof. See Appendix for proof. ■

The proposition above translates to the fact that firms become non-active when targets are achieved by simply charging a zero rate and thus, substitute for rates charged of active firms in such a way that the outcome is the same or is greater than the outcome from the non-active firms' autarkic rate charged to debtors. It also holds then that Nash equilibrium for the game has to include atleast one active firms such that A cannot be a null set. We show a simple algorithm in the appendix which efficiently computes this equilibrium[5]. Furthermore, we draw the following statement from the proposition 2 as follows;

[5] A simple computational algorithm takes $2^{|\mathcal{N}|} - 1$ iterations representing possible combination of active firms. It is noteworthy that even if we relax assumption 2 so that $\hat{\mathcal{N}} = \{\mathcal{N}, \mathcal{S}\}$, Equilibrium is simply obtainable by computing for $\mathcal{N}$. Hence, for each firm $i \in \mathcal{N}$ such that a firm $j \in \mathcal{S} \cap \mathcal{N}^{out}_i$, then $b_{-i} = b_{ij} + \ldots$

Corollary 1. *Assume that* $\forall\ i,j \in \mathcal{A}$, $b_{-i} = b_{-j}$ *so that* $\boldsymbol{\pi} = \pi \boldsymbol{\cdot} \mathbf{1}_{\mathcal{A}}$. *This means that* $\mathbf{r}^*(\mathcal{A}, -a) = \pi \boldsymbol{\cdot} \boldsymbol{b}(\mathcal{A}, -a)$ *so that* $\forall$ *firm* $i \in \mathcal{A}$*;*

$$r_i^*(\mathbf{G}, \mathcal{A}, a) = \pi \boldsymbol{\cdot} \beta_i(\mathcal{A}, -a),$$

where $\beta_i(\mathcal{A}, -a)$ *refers the Bonacich independence index*[6] *or simply independence index of an active firm* i *implying* $\boldsymbol{b}(\mathcal{A}, -a) = (\beta_i(\mathcal{A}, -a))_{i\in\mathcal{A}} \in \mathbb{R}_+^n$.

Proof. Because we have the following;

$$\boldsymbol{b}(\mathcal{A}, -a) \stackrel{\text{def}}{=} (\mathbf{I} + a\mathbf{G})^{-1}_{\mathcal{A}\times\mathcal{A}} \boldsymbol{\cdot} \mathbf{1}_{\mathcal{A}}. \tag{3.2}$$

■

This implies that Nash equilibrium rate is of each firm is directly proportional to their independence index. The independence index is so named because $\mathbf{G} = [g_{ji}]$ accounts for the strength of incoming links. Also since in the series, $(\mathbf{I} - a\mathbf{G})_{\mathcal{A}\times\mathcal{A}} \boldsymbol{\cdot} \mathbf{1}_{\mathcal{A}}$ dominates $((\mathbf{I} - a\mathbf{G})_{\mathcal{A}\times\mathcal{A}} \boldsymbol{\cdot} \boldsymbol{\pi}_{\mathcal{A}}$ dominates the Nash equilibrium $\mathbf{r}(\mathcal{A}, -a))$, then the greater the strength of g_{ji} for each firm i, the lower its $\beta_i(\mathcal{A}, -a)]$. This then hints as to which firm i charging less amount in lending rate. We explore some special network properties in relation to this in the next section.

3.3 Equilibrium and Inactive Firms

Our proposition 2 shows that Nash equilibrium could be such that $\mathcal{N} - \mathcal{A} \neq \{\}$. A firm $i \in \mathcal{N} - \mathcal{A}$ thus has an $r_i = 0$ as its equilibrium rate charged to its debtors. We draw a swift distinction between inactive firms in our model and the concept of *free-riders* found in major public goods in networks papers such as Bramoullé and Kranton (2007), Bramoullé et al. (2014) as well as Allouch (2015). To understand this is to understand the best replies given in (2.8) as an outcome of the payoff. Observe that loan management is a main objective of the firm and as such, strategic substitution arises in a bid to reduce such management cost. So while a firm who borrows cannot influence (directly) the rate

[6] So as not to confuse it with Bonacich Centrality which is $\beta_i(\mathbf{G}^T, a)$ for a firm i.

to which it is charged, it can charge a corresponding rate to its debtors to balance and optimize loan management expenses. For this reason, charging a zero rate to debtor thus arises from the fact the present loan management cost is quite substantial that a positive rate would be even more harmful to the firm.

The idea here is that an inactive firm $i \in \mathcal{N} - \mathcal{A}$ is not necessarily *free-riding* the provision of other firms but on the other hand, is simply avoiding any further cost as a result of its own decision since its creditors has increased such cost to the maximum.

4 Core-Periphery Networks

We explore in a unique way, further properties of our equilibrium. More specifically we aim to discuss and show unique network properties of active firms and what the implication might be in terms of the equilibrium lending rate each firm charges.

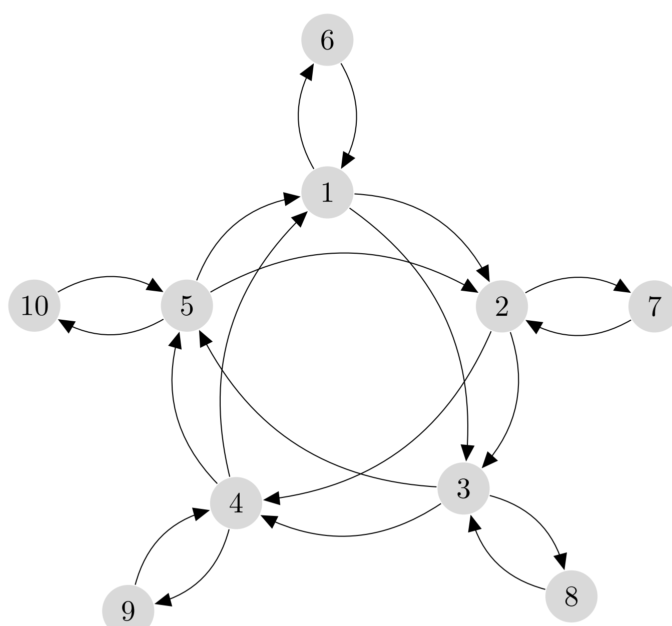


Figure 3: Core-periphery network bilateral links between periphery and core sets.

To do this, we examine a stylized case of a network which is shown in fig. 5 which contains a core-periphery network where each firm in the core-periphery lends and borrows to others. Core-periphery network has been widely stylized within the inter-bank network literature especially within the line of financial contagion and systemic risk. Recent examples of such studies include Chiu, Eisenschmidt, and Monnet (2020), Lux, Fricke, et al. (2012), Van Lelyveld et al. (2014) as well as Sui, Tanna, and Zhou (2020). We describe a core-periphery network as one which has 2 groups of firms, the core firms whose set we denote as Cr and the peirphery set which we denote as Pr. For a digraph $\mathbf{g}(\mathcal{N}, g)$ such that $\Omega = \{Cr, Pr\} = \mathcal{N}$. Also, assume $\mathcal{N}(\Omega, \mathbf{r}^*) = \mathcal{A}$ such that all firms in the core-periphery network are actively charging lending rates at equilibrium. The core-periphery network has the following graph form;

$$\mathbf{G}(Cr, Pr) = \left[\begin{array}{c|c} Cr \times Cr & Cr \times Pr \\ \hline Pr \times Cr & Pr \times Pr \end{array}\right] = \left[\begin{array}{c|c} \mathbf{G(CC)} & \mathbf{G(CP)} \\ \hline \mathbf{G(PC)} & \mathbf{G(PP)} \end{array}\right] \tag{4.1}$$

For a network to be deemed core-periphery, it means it can be grouped into the block partition as shown above.

Assumption 3 (Block Matrix Properties)**.** *Given $\mathbf{G}(Cr, Pr)$ which is strictly unidirectional let $\Omega = \{Cr, Pr\}$. We have the following conditions;*

1. $|Pr| = |Cr|$,
2. $\mathbf{G(PP)} = \mathbf{0}$,
3. $\mathbf{G(PC)} = \varrho \bullet \mathbf{I}$,
4. $\mathbf{G(CP)} = \theta \bullet \mathbf{I}$.

This indicates that we allow for bilateral relationships.[7]. We are able to compute the Nash Equilibrium using the block-partition matrix as in (4.1) to compute each firms Nash Equilibrium:

Proposition 3. *Let $\mathbf{g}(\Omega, g)$ be a directional graph whose topology is core-periphery in nature. Also assume $\Omega = \{Cr, Pr\}$, $\boldsymbol{\pi} = (\boldsymbol{\pi}_{Cr}, \boldsymbol{\pi}_{Pr})^T$ and assumption 3 holds. In so far*

[7] One could possibly argue that bilateral liabilities would not hold given that it can simply be netted off. However, because each firm makes separate lending rate decision, bilateral links need be exactly as they are as contractual properties might differ as we show for example in fig. 5 where periphery and core have between them, such bilateral relationships.

as $a \in \left]0, \frac{1}{\left|\nu_{min}(\frac{\mathbf{G}+\mathbf{G}^T}{2})\right|}\right[$, *we have the following Nash Equilibrium;*

$$\mathbf{r}^*_{Cr}(\mathbf{G}, \Omega, a) = \left(\mathbf{I} + \frac{a}{1-a^2\theta\varrho}\mathbf{G}(\mathbf{CC})\right)^{-1} \frac{(\boldsymbol{\pi}_{Cr} - a\theta\boldsymbol{\pi}_{Pr})}{1-a^2\theta\varrho}, \tag{4.2}$$

$$\mathbf{r}^*_{Pr}(\mathbf{G}, \Omega, a) = \boldsymbol{\pi}_{Pr} - a\varrho \cdot \mathbf{r}^*_{Cr}(\mathbf{G}, \Omega, a). \tag{4.3}$$

Proof. See Appendix for proof. ■

The (4.3) above then provides us with an initial intuition as we can see that for any firm $i \in Cr$ and a firm $j \in Pr | j \in \{\mathcal{N}_i \cap Cr\}$, then it holds that;

$$a g_{ij} \cdot r^*_i(\mathbf{G}, \Omega, a) + r^*_j(\mathbf{G}, \Omega, a) = \pi_j, \tag{4.4}$$

thus implying a direct strategic substitution relationship between Nash lending rate decision of each core and its corresponding periphery. The greater the Nash rate the core charges, the less its corresponding periphery lending rate is and vice versa.

Furthermore, (4.2) shows the core set Nash Equilibrium is then modified into a measure which includes the value $\frac{a}{1-a^2\theta\varrho}$. This parameter takes the form of a *new attenuation parameter* such that it replaces the initial attenuation parameter a. The new attenuation parameter is greater as $\frac{a}{1-a^2\theta\varrho} > a$ and since the Bonacich expression in (4.2) takes the power series $\mathbf{I} - \frac{a}{1-a^2\theta\varrho}\mathbf{G}(\mathbf{CC}) + \left(-\frac{a}{1-a^2\theta\varrho}\mathbf{G}(\mathbf{CC})\right)^2 + \left(-\frac{a}{1-a^2\theta\varrho}\mathbf{G}(\mathbf{CC})\right)^3 + \ldots$, it implies that greater substitution from distant neighbours. However, this time, the weight of relationship between the core and periphery set of firms now determines how much of such weight is accounted for in the Nash equilibrium . To show, since we have that $1 - a^2\theta\varrho \downarrow$ if either θ or ϱ rises, then it means that the link $\frac{a}{1-a^2\theta\varrho}\mathbf{G}(\mathbf{CC})$ is strengthened to such rise in θ and/or ϱ and weakened when the reverse holds.

Lastly, the density of the core set is accounted for in the matrix $\mathbf{G}(\mathbf{CC})$ such that the diversification of core firms within the core network is crucial in determining the rate to which the core firms would charge.

4.1 Equitable Partition

We are able to understand further special properties of the core periphery relation under possible stylized partition network property. We hold the following assumption for this part of the paper;

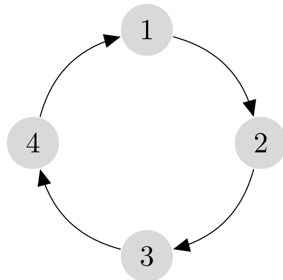
Assumption 4. *Given* $\Omega = \{Cr, Pr\}$*, we hold that* $\mathbf{G}(\mathbf{CC}) \bullet \mathbf{1} = \rho \bullet \mathbf{1}$*, while* $\mathbf{G}(\mathbf{PC}) = \varrho \bullet \mathbf{I}$ *and* $\mathbf{G}(\mathbf{CP}) = \theta \bullet \mathbf{I}$ *for* $\rho, \varrho, \theta \in \,]0, \mathbb{R}_{++}]$*.*

This yields a core-periphery network which has both out-equitable and in-equitable properties as defined using Kada (2020) as well as Deng, Sato, and Wu (2007) as follows;

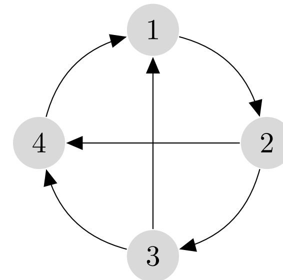
Definition 5 (Equitable Partition)**.** *Consider* $\Omega = \{Cr, Pr\}$ *where* $\mathbf{G}(\mathbf{CC}) \bullet \mathbf{1} = \rho \bullet \mathbf{1}$ *and* $\mathbf{G}(\mathbf{PP}) \bullet \mathbf{1} = 0 \bullet \mathbf{1}$ *arising from* $\mathbf{G}(\mathbf{PP}) = \mathbf{0}$ *(from assumption 3) so that* Cr *and* Pr *are* Partitions*. If we have that* $\mathbf{G}(\mathbf{PC}) = \varrho \bullet \mathbf{I}$ *and* $\mathbf{G}(\mathbf{CP}) = 0 \bullet \mathbf{I}$ *then* Ω *is 'out-equitable' while if we have that* $\mathbf{G}(\mathbf{CP}) = \theta \bullet \mathbf{I}$ *and* $\mathbf{G}(\mathbf{PC}) = 0 \bullet \mathbf{I}$*, then* Ω *is 'in-equitable'. Where both* $\mathbf{G}(\mathbf{PC}) = \varrho \bullet \mathbf{I}$ *and* $\mathbf{G}(\mathbf{CP}) = \theta \bullet \mathbf{I}$ *holds simultaneously then* Ω *is simply an 'Equitable' partition.*

Since core network are directed ring-network, observe the examples as in fig. 5a as well as a complete bi-directional core-network shown in fig. 6b. Also from the fig. 6 , observe that all but fig. 5b are partitioned cores[8] as each core-firm has same amount of incoming and outgoing link in each case.

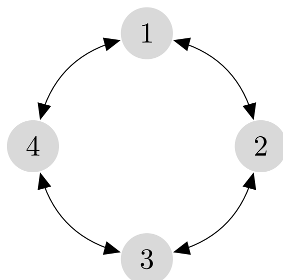
[8]For example, $\rho = 1$ for fig. 5a, $\rho = 2$ in fig. 6a while $\rho = 3$ in fig. 6b. Because $\mathbf{G}(\mathbf{CC}) \bullet \mathbf{1} = (2, 1, 1, 2)^T$ for fig. 5b, we are unable to define ρ in such case.



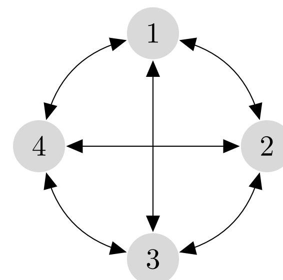
(a) Directed-Ring Core Network.



(b) Irregular Core Network.



(c) Bilateral-Ring Core Network.



(d) Complete Core Network.

Figure 4: Sample Core Networks. The figure a, c and d involve cores with fit the partition criteria.

We begin with an added assumption which satisfies both the ring core-network, core network with regular ring properties based on proposition 3 to show some realization in the following statement below;

Proposition 4. *Let* $\mathbf{G}$ *bi-directional graph of SCF which are core-periphery in nature as in* $\Omega = \{Cr, Pr\}$ *and assumption 4 holds. In so far as* $'a'$ *is within threshold, we have the following Nash Equilibrium;*

$$\begin{aligned} \mathbf{r}^*_{Cr}(\mathbf{G}, \Omega, a) &= (1 + a\rho - a^2\theta\varrho)^{-1}(\boldsymbol{\pi}_{Cr} - a\theta\boldsymbol{\pi}_{Pr}), \\ \mathbf{r}^*_{Pr}(\mathbf{G}, \Omega, a) &= \boldsymbol{\pi}_{Pr} - a\varrho \bullet \mathbf{r}^*_{Cr}(\mathbf{G}, \Omega, a), \end{aligned} \tag{4.5}$$

Proof. See Appendix for proof. ■

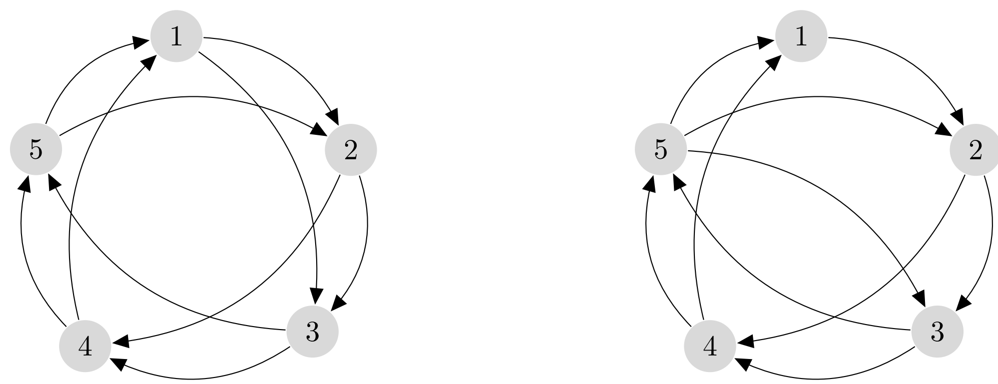
This is such that if $\boldsymbol{\pi} = \pi \bullet \mathbf{1}$, each within a partition charges identical lending rates. Observe further intuitions from this proposition,

Remark 4.1. *If* $\boldsymbol{\pi} = \pi \bullet \mathbf{1}$ *and* $\theta = \varrho$*, the expression* $r^*_{Pr}(\mathbf{G}, \Omega, a) > r^*_{Cr}(\mathbf{G}, \Omega, a)$ *always holds true in so far* $\rho > 0$.

The remark above points to the fact that if a borrowing network meets the criteria for core-periphery relationship where links between each periphery and its core are identical and each firm lends same total amount, then one can presume core firms would charge a lower lending rate as compared to the peripheries.

To discuss more on the condition that $\mathbf{G}(\mathbf{CC})\boldsymbol{\cdot}\mathbf{1} = \rho\boldsymbol{\cdot}\mathbf{1}$ in assumption 4, we illustrate this in the examples below;

Example 1. Assuming the following networks with homogeneous links such that each edge is weighted $\alpha \in]0, \mathbb{R}^{+}[$ below;



(a) Regular Core Network *(b) Irregular Core Network*

Figure 5: Core Network with homogeneous links.

We have the sub-matrix of core interconnections as;

$$\mathbf{G}_1(\mathbf{CC}) = \alpha \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}, and \qquad \mathbf{G}_2(\mathbf{CC}) = \alpha \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{G}_1(\mathbf{CC})\boldsymbol{\cdot}\mathbf{1}_{|Cr|} = \mathbf{G}_2(\mathbf{CC})\boldsymbol{\cdot}\mathbf{1}_{|Cr|} = \alpha(2,2,2,2,2)^T$$

This implies that,

$$\mathbf{G}_1(\mathbf{CC})/\Omega = \mathbf{G}_2(\mathbf{CC})/\Omega = \rho = \alpha\boldsymbol{\cdot}2$$

Example 2. Let us take another set of networks, this time with heterogeneous links as follows;

(a) Directed Core Network. *(b) Directed Core Network.*

Figure 6: Core Networks with heterogeneous links.

We have the sub-matrix of core interconnections as;

$$\mathbf{G}_1(\mathbf{CC}) = \begin{bmatrix} 0 & 0 & 0.4 & 0.4 \\ 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \end{bmatrix}, and \qquad \mathbf{G}_2(\mathbf{CC}) = \begin{bmatrix} 0 & 0 & 0.4 & 0.8 \\ 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}$$

$$\mathbf{G}_1(\mathbf{CC}) \cdot \mathbf{1}_{|Cr|} = (0.8, 0.8, 0.8, 0.8)^T,$$
$$\mathbf{G}_2(\mathbf{CC}) \cdot \mathbf{1}_{|Cr|} = (0.8, 0.8, 1.2, 0.4)^T.$$

This implies that,

$$\mathbf{G}_1(\mathbf{CC})/\Omega = \rho = 0.8$$
$$\mathbf{G}_2(\mathbf{CC})/\Omega = ??$$

As such the core network in fig. 6b does not satisfy the assumption 4.

5 Intervention and Welfare Policies

In this section, we define outcomes based on Nash lending rates and then observe welfare properties of the model. More precisely, we highlight various possible policy initiative to which a maximising planner could adopt and its estimate the overall impact. To study welfare, we adopt the standard utilitarian approach. As such, we introduce the following definition;

Definition 6. *The welfare from the game* $\Gamma(\mathbf{G}, \mathbf{r}, a)$ *is defined specially for firms who charge a positive amount as;*

$$W(\mathbf{r}, \mathcal{A}, a) \stackrel{\text{def}}{=} \sum_{i \in \mathcal{A}} P_i, \tag{5.1}$$

This implies we use welfare is the aggregate payoff of all firms who charge a positive amount. To define such payoff, we write the following lemma;

Lemma 2. *Assume* $\mathcal{N}$ *and the game* $\Gamma(\mathbf{G}, \mathbf{r}, a)$, $\forall$ *firm* $i \in \mathcal{A}$, *payoff given Nash equilibrium is as follows;*

$$\mathbf{P}_{\mathcal{A}} = diag(\mathbf{B}) \cdot \left((\mathbf{I} + a\mathbf{G})^{-1} \cdot \boldsymbol{\pi}_{\mathcal{A}} \right) - \boldsymbol{K}, \tag{5.2}$$

where $\boldsymbol{K} = \left(\frac{2+a}{4\kappa a}\right) \in \mathbb{R}_+^{|\mathcal{A}|}$ *and* $\mathbf{B} = (\alpha \cdot b_{-i})_{i \in \mathcal{A}} \in \mathbb{R}_+^{|\mathcal{A}|}$ *are both column vectors.*

Proof. See Appendix for proof. ■

It is important to note the implication of (5.2). We see here that firms utility for charging is mainly dependent on their individual Nash equilibrium rate. This means that if we were to observe (5.2) and our best reply in (2.5) we then have an idea of kinds of policy implications for the model which we explore in the coming sections.

5.1 Transaction Cost and Welfare Neutrality

In this part, we explore the possibility of intervention policies and their welfare impact. Usually in payment systems, movement of a cash could face barriers such as foreign exchange conversion cost (if 2 firms are located at different economic regions), transaction cost like bank charges, etc. If we assume a system where firms incur transaction cost on total payment which we denote as λ, let us have a case where a regulator decides to grant λb_{-i} to each $i \in \mathcal{N}$, such policies are done so far as they keep the active set $\mathcal{A}$ the same which means the network graph $\mathbf{G}_{\mathcal{A}}$ should remain unchanged, unchanged. Given λb_{-i} for all firm $i \in \mathcal{N}$. The initial mark of the policy λ is such that payoff is written as;

$$P_i^{\lambda}(r_i) = \lambda \left(b_{-i} \bullet r_i - \sum_{j \in \mathcal{N}_i^{in}} (\hat{g}_{ji} b_i)\, r_j \right) - \kappa \left(\lambda \mu_i\right)^2 \tag{5.3}$$

The diagram fig. 7 shows a planner P with a linear problem such that $\beta(\bullet) > c(\bullet)$. More specifically, the fig. 7a represents and instance where a planner increases payment made by each firm to another (for example, through elimination of a prevailing transaction cost like bank charges) while the fig. 7b shows a case where even without friction, the planner grants each lender an extra amount to loan its potential debtors given strictly that debtors are not opposed to such additional loans. The arrows show the policy action. In terms of equilibrium, we introduce the following lemma;

Lemma 3. *The active set $\mathcal{A}$ remains fixed $\forall\ \lambda \lesseqgtr 1$ even though $\mathbf{r}^{\lambda} = \lambda^{-1} \bullet \mathbf{r}$.*

Proof. For the Nash equilibrium given such policies, we have for all active firms that;

$$r_i^{\lambda} = \frac{\pi_i}{\lambda} - a \sum_j \frac{\lambda b_{ji}}{\lambda b_{-i}} r_j^{\lambda} = \frac{\pi_i}{\lambda} - a \sum_j \frac{b_{ji}}{b_{-i}} r_j^{\lambda}. \tag{5.4}$$

This is so that rewriting in vector form, our Nash for active firms is given as;

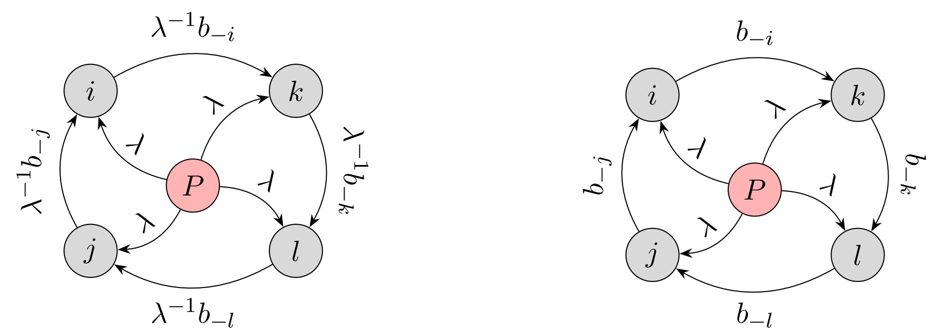
$$\mathbf{r}^{\lambda} = (\mathbf{I} + a\mathbf{G}_{\mathcal{A}})^{-1} \bullet \frac{\boldsymbol{\pi}_{\mathcal{A}}}{\lambda} = \lambda^{-1} \bullet \mathbf{r}.$$

Given the Bonacich centrality element of Nash $(\mathbf{I}+a\mathbf{G}_{\mathcal{A}})^{-1}$ remains fixed in λ, we hold that class of active firms given a substitute game $\mathcal{A}$ changes if and only if the proportionality of $\boldsymbol{\pi}$ is altered. However, $\frac{1}{\lambda}\boldsymbol{\pi} \propto \boldsymbol{\pi}$ as such implying set of Active firms $\mathcal{A}$ are fixed for $\lambda \lesseqgtr 1$ since λ^{-1} is a constant. ∎

Some explanation of this lemma is that since transaction cost λ is homogeneous and since inactive firms are such that $a\mathbf{G}_{\mathcal{N}-\mathcal{A}\times\mathcal{A}} \bullet \mathbf{r}_{\mathcal{A}}^{\lambda} \geq \boldsymbol{\pi}_{\mathcal{N}-\mathcal{A}}^{\lambda}$, then it means that while it is that $r_{i\in\mathcal{A}}^{\lambda} = \lambda^{-1} r_i$, it is also the case that $\pi_{i\in\mathcal{N}-\mathcal{A}}^{\lambda} = \lambda \pi_i$. So if because π rises and falls at equal magnitude for each firm, set of active firms remains constant. As such the magnitude of transaction cost or intermediate intervention is not relevant in terms of what the composition of active set would be at Nash equilibrium. In that light, we summarise the effect of such homogeneous intervention policy as follows;

Proposition 5. *Given the homogeneous policy λ, $\Delta W(\mathbf{r}^{\lambda}, \mathcal{A}, a) = 0$, hence welfare is neutral.*

Proof. See Appendix for proof. ■



(a) Intervention in a system with existing frictions.

(b) Intervention in a system without existing frictions.

Figure 7: Directions of a planner P's intervention to 4 firms

We then move to observe the impact of mutually exclusive policy $\lambda_i b_{-i}\ \forall\ i \in \mathcal{A}$ such that $\lambda_i \lesseqqgtr \lambda_j$ for all $i, j \in \mathcal{A}$. Observe here that policies are restricted to active set, we assume strictly that such policy intervention is such that leaves active set unchanged. For simplicity, one can initially assume the policy λ_i is applied to a single firm while holding others fixed as shown in fig. 8 where this time a regulator increases only one firms borrowing . In practice, it could be through eliminating transaction cost for a single firm while leaving other constant as shown in the figure below;

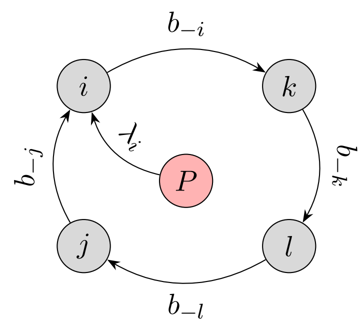


Figure 8: Ring network with 4 firms where a regulator decides to increase total firm i's lending to $\lambda_i b_{-i}$.

More broadly, the concept of the policy is that links of firms could be increased at heterogeneous proportion. The impact of such policy on welfare goes as follows;

Theorem 1. *Given a policy $(\lambda_i, \lambda_j, \ldots)$ so that $\lambda_i \lesseqqgtr \lambda_j$ for all $i, j \in \mathcal{A}$ we have the following outcome;*

$$\Delta W^{\lambda}(\mathbf{r}^{\lambda}, \mathcal{A}, a) = 0 \tag{5.5}$$

Hence such policy is welfare neutral.

Proof. See Appendix for proof. ■

This means that it is not possible for a regulator to improve the welfare of active players by simply increasing/reducing one or more active firm network intensity even if it is by varying amounts. Welfare Neutral policies are also found in major public literature such as Bergstrom et al. (1986) and Warr (1983) who both showed neutrality to aggregate provision of public good and individual consumption of private good in so far as wealth redistribution does not change the set of active players involved. In an extension to this, Allouch (2015) adds that small transfers that leave active set the same are also neutral only when such transfers are made between the active set themselves. To contrast with our results yield neutrality without transfer policies. Because each firms utility is based on their individual Nash equilibrium, payoffs are neutral which leaves overall welfare unchanged. Additionally, intervention are not be restricted to active firms and due to the homogeneous nature of intervention, the magnitude of λ is pertinent in influencing the outcome in so far rates charged by creditor firms are limited to non-negative rates.

5.2 Resource Allocation

To access a possible impact of theorem 1, we observe a policy change of $\Delta\ \kappa_i$ for $i \in \mathcal{A}$ (i.e, firms who charge at Nash equilibrium). Let us have the following definition;

Definition 7. *For any firm $i \in \mathcal{N}$, we have it that*

- *Subsidy* $\Rightarrow \frac{\Delta\kappa_i}{\kappa_i} = \gamma_i^-$
- *Tax burden* $\Rightarrow \frac{\Delta\kappa_i}{\kappa_i} = \gamma_i^+$

Assume then that $\gamma_i = \gamma\ \forall$ firm $i \in \mathcal{A}$ so that the policy is applied in homogeneous proportion to all active firms. Payoff of each firm i is written as;

$$\forall\ i \in \mathcal{N} \quad P_i^\gamma(r_i) = b_{-i} \bullet r_i - \sum_{j \in \mathcal{N}_i^{in}} (\hat{g}_{ji} b_i)\, r_j - (1+\gamma)\kappa(\mu_i)^2 \tag{5.6}$$

We summarise the effect in the following results;

Lemma 4. *Given γ, welfare differential is as follows;*

$$\Delta W^\gamma(\mathbf{r}^\gamma, \mathcal{A}, a) = \mathbf{1}^T \mathbf{P}_{\mathcal{A}} \bullet \frac{-\gamma}{(1+\gamma)}. \tag{5.7}$$

Remark 5.1. *This implies that if $\gamma \in [-1, 0[$, then $\Delta W^\gamma(\mathbf{r}^\gamma, \mathcal{A}, a) > 0$ while if $\gamma \in [0, 1[$ then $\Delta W^\gamma(\mathbf{r}^\gamma, \mathcal{A}, a) < 0$ and its interpretation is simply that subsidies improves welfare while taxes reduce welfare.*

Note that Active firms $\mathcal{A}$ also remains fixed $\forall\ \gamma \in\ ]0, 1[$. Results in this case are clearly unsurprising as lighter burden means firms are less sensitive to the volume of indebtedness given its fixed debt. Examples of such policies could be through providing outsourcing facility to a portion of debts or maybe policies to reduce call rates or providing free training of labour force involved in such area. When however, this policy applies in a heterogeneous manner to firms, it then becomes isomorphic to resource transfers which we explore in details subsequently.

In lemma 3 as well as theorem 1, it is noted that given a policy $\lambda_i \lesseqgtr 1$ such that lending becomes $\lambda_i b_{-1}$ for any $i \in \mathcal{A}$, $\Delta W(\mathbf{G}, a) = 0$ in so far as the active firms $\mathcal{A}$ remains fixed. Given our results above, we have the following results;

Proposition 6. *Given the game* $\Gamma(\mathbf{G}, \mathcal{A}, a)$ *there exists* $\Delta W(\mathbf{r}, \mathcal{A}, \gamma, \lambda) \in\,]0, \mathbb{R}_{++}[$ *(not necessarily Pareto) at zero cost to a planner in so far as there exists* $\sum_{i \in \mathcal{A}} \lambda_i b_{-i}$ *such that* $\mathcal{A}$ *remains fixed.*

Proof. Strictly holding $\mathcal{A}$ fixed, let $\sum_{i \in \mathcal{A}}(1-\lambda_i)b_{-1}$ be the amount the regulator charges for intermediate payments from each firm i (building from proposition 5) , then this is the case so far $\sum_{i \in \mathcal{A}}(1-\lambda_i)b_{-1} = \gamma \sum_{i \in \mathcal{A}} (\kappa_i(\mu_i)^2)$ which then guarantees Pareto improvement among active firms. For non-Pareto improvement, subsidised administrative cost $\gamma_i \kappa_i(\bullet)^2$ need not apply to all firms in $\mathcal{A}$. In this case, the criteria shown in theorem 2 becomes useful. ■

This comes form the fact that so far as active set remains fixed, the regulator can instead of eliminating transaction cost, create one at no cost to overall welfare. This also grants resources to subsidise one or more firms in a way that improves welfare. Pareto improvement is possible if $X = \sum_{i \in \mathcal{A}}(1-\lambda_i)b_{-1}$ is split such that $\gamma \sum_{i \in \mathcal{A}} \kappa_i(\mu_i)^2 \leq X$. Observe now that γ is constant so that its effect on welfare corresponds to lemma 4. This is a unique form of transfer compared to those found in mainstream public good in networks literature such as Allouch (2015), Allouch and King (2018b), etc. This is because in this case, transfers could be simply from one firm to another through different variables the firm faces.

6 Intervention Targeting

We project in this section the relationship between Bonacich externality measures and firms quality, especially in terms of marginal welfare given a resource constrained planner. We here generalise the Planner to one who wishes to grant loan management cost subsidy in order to maximise overall welfare $(\Delta W^{\gamma}(\mathbf{r}^{\gamma}, \mathcal{A}, a))_{max}$ of active firms $\mathcal{A}$. Then if the set $\Phi(\mathcal{A})$ represents the possible combinations of firms, the planner has $|\Phi(\mathcal{A})| = 2^{|\mathcal{A}|} - 1$ amount of alternative actions as to the distribution of subsidy intervention in order to achieve $(\Delta W^{\gamma}(\mathbf{r}^{\gamma}, \mathcal{A}, a))_{max}$. This is such that the eariler discussed "γ $\forall$ firm $i \in \mathcal{A}$" is a strategy element in $\Phi(\mathcal{A})$ arising from the $C(|\mathcal{A}|, |\mathcal{A}|)$ combination, where $C(a, b) = \frac{a!}{(a-b)!b!}$. On the other extreme, let $\phi \subset \Phi$ be the subset arising the combination $C(|\mathcal{A}|, 1)$, This then means that $|\phi| = |\mathcal{A}|$ such that the Planner calculates the total welfare from subsidising for a single firm $i \in \mathcal{A}$. We then wish to show the qualities of the firm $i \in \mathcal{A}$ which yields the greatest payoff from the strategy subset ϕ. Literature in recent times have, within network spillover problems come up with various targeting criterion; The *Key-Player* concept introduced in Ballester et al. (2006), The highest threat index (which is the Bonacich centrality) introduced in Demange (2016) as well as the *top Principal Components* as another eigenvalue related measure used in Galeotti et al. (2020).

We begin with a naive scenario. Assume a planner with unlimited finance but one who wishes to subsidise administrative cost by a $\gamma\%$ for a single selected firm so as to maximise overall network welfare. Formally, we define the planners problem within the strategy $\phi \subset \Phi$ is stated as;

$$\max_{\gamma} \{P_i^{\gamma} - P_i | i = 1, \ldots, n\} \qquad s.t \quad \gamma^{-} = \gamma_i | i \in \{\mathcal{A}\}. \tag{6.1}$$

The choice firm $i \in \mathcal{A}$ then has a payoff is written as;

$$P_i^{\gamma}(r_i) = \left(b_{-i} \bullet r_i - \sum_{j \in \mathcal{N}_i^{in}} (\hat{g}_{ji} b_i)\, r_j \right) - (1+\gamma)\kappa(\mu_i)^2 \tag{6.2}$$

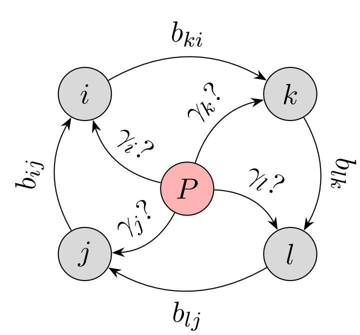


Figure 9: Ring network with 4 firms to which the planner makes a decision which to subsidise.

Hence the question is which firm should the planner subsidise for? Observe the following equation of the measure of a firm $i \in \mathcal{N}$;[9]

[9]We still hold in this part that $\mathcal{N} = \mathcal{A}$.

$$\beta_i(\mathbf{G}^T, -a) \overset{\text{def}}{=} \sum_{k=0}^{+\infty}(-a)^k \sum_{j=1}^{n} \left(\left(\mathbf{G}^T\right)^k \right)_{ij} \tag{6.3}$$

This is such that $\boldsymbol{b}(\mathbf{G}^T, -a) = (\mathbf{I} + a\mathbf{G}^T)^{-1} \cdot \mathbf{1} = \left(\beta_i(\mathbf{G}^T, -a)\right)_{i \in \mathcal{N}} \in \mathbb{R}^n_+$. The measure above is related to the Bonacich centrality used to capture prestige and network influence as proposed by Bonacich (1987). However, it measures the weakness of firms link to its debtors. This means that the greater $\beta_i(\mathbf{G}^T, -a)$ is for a firm i, the smaller the weight of the direct link to $\mathcal{N}_i^{out}$. Going further, $\beta_i(\mathbf{G}^T, -a)$ is referred to as the *externality index* for firm i. We as such present the following results.

Theorem 2. *Assume that $b_{-i} = b_{-j}$ $\forall$ $i, j \in \mathcal{A}$. The welfare differential $\Delta W(\mathbf{r}^\gamma, \mathcal{A})$ is at maximum if and only if subsidy γ_i such that for firm i ;*

$$\beta_i(\mathbf{G}^T_{\mathcal{A}}, -a) \geq \beta_{j \neq i}(\mathbf{G}^T_{\mathcal{A}}, -a),$$

Hence firm i has the largest externality index.

Proof. See Appendix for proof. ■

This result shows the relationship between externalities on outgoing links based on weighted interconnections and ability to improve overall welfare overall from intervention related to subsidy. To summarise this point, recall that we can also write firm i's centrality measure as below,

$$\beta_i(\mathbf{G}^T_{\mathcal{A}}, -a) = 1 - a \sum_{j \in \mathcal{N}_i^{out}} g_{ij} \beta_j(\mathbf{G}^T_{\mathcal{A}}, -a). \tag{6.4}$$

This means that for every unit increase in π_i, it negatively impacts each $r_{j \in \{\mathcal{N}_i^{out} \cap \mathcal{A}\}}$. Thus a negative externality. Then given that lending rates charged by active firm serve as a form on negative externality, the subsidy should be given to the firm who produces the least externality in the network. This is as subsidy here increases strategic substitution since it increases the potential r_i for any firm whose $\kappa(\mu_i)^2$ is reduced. This serves as an identifier for *pressure points* of our model in contrast to other network targeting works.

A more practical and justifiable scenario would be where the planner has limited resource. In this instance, the planner wishes to maximise total welfare and as such, measures the impact of channeling subsidy to a single firm versus splitting proportionally across all active firms. In order to select the firm to consider allocating resource to, let us rewrite the problem of the planner from (6.1) as follows;

$$\max_{\gamma_i|i\in\mathcal{A}} \left\{P_i^{\gamma_i} - P_i | i = 1, \ldots, n\right\}, \tag{6.5}$$

$$s.t \quad \gamma_i = \gamma_i | i \in \{\mathcal{A}\} \quad and,$$
$$\gamma_i \bullet \kappa_i(\mu_i)^2 \leq X$$

It follows then that $\gamma_i \leq -\frac{X}{\kappa_i(\mu_i)^2}$ where X represents the cash endowment of the regulator. In this case, we then derive another corollary from theorem 2 as,

Corollary 2. *Assuming a regulator who is cash constrained and $b_{-i} = b_{-j}\ \forall\ i, j \in \mathcal{A}$, the welfare differential $\Delta W^{\gamma_i}(\mathbf{r}^{\gamma_i}, \mathcal{A}, a)|i \in \mathcal{A}$ is at maximum if and only if subsidy γ is applied to firm i which meets the following criteria,*

$$\beta_i(\mathbf{G}_{\mathcal{A}}^T, -a) \bullet \frac{-\gamma_i}{1+\gamma_i} \geq \beta_{j\neq i}(\mathbf{G}_{\mathcal{A}}^T, -a) \bullet \frac{-\gamma_j}{1+\gamma_j}.$$

Proof. Since γ_i is not necessarily homogeneous across firms, then $\forall$ firm i such that $P_i = \ldots + (1+\gamma_i) \bullet \kappa(\mu_i)^2$, $\Delta W(\mathbf{r}^{\gamma}, \mathcal{A}, a) = \frac{-\gamma_i \alpha}{2\kappa(1+\gamma_i)} \bullet \beta_i(\mathbf{G}_{\mathcal{A}}^T, -a) + \frac{\eta\gamma_i}{1+\gamma_i} = \frac{\gamma_i}{1+\gamma_i}\left(\eta - \frac{\alpha}{2\kappa} \bullet \beta_i(\mathbf{G}_{\mathcal{A}}^T, -a)\right)$ and we also hold that $\frac{\gamma_i}{1+\gamma_i} \to +\infty$ as $\gamma_i \to -1$ while keeping active set $\mathcal{A}$ strictly fixed. ■

The intuition then from our results is that welfare due to individual subsidy especially when the regulator has limited funds are best allocated to firms with a combination of greater proportional reduction in loan management expenses as well as lower negative spillover effects. An example of the planner making this decision can be observed below;

Example 3 (Individual vs Group Targeting)**.** Assuming the following debt network below;

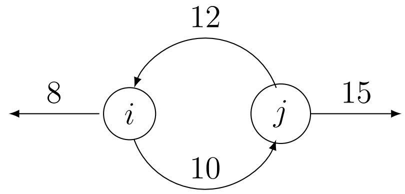


Figure 10: Network with 3 firms and 4 debt contracts (edges)

Other parameters are as follows, $a = 0.8$, $\kappa = 0.04$. This means we have $\boldsymbol{\pi} = (0.699, 0.46)^T$ and

$$\mathbf{G} = \begin{bmatrix} 0 & 0.67 \\ 0.37 & 0 \end{bmatrix}.$$

So that $\mathbf{r}^* = (0.54, 0.3)^T$, $\mathbf{b}(\mathbf{G}^T, -a) = (0.8368, 0.5515)^T$ and $\mathbf{P} = (19.198, 15.553)^T$. Which leaves the initial welfare $\mathbf{1}^T\mathbf{P} = 34.751$.

Assume then that a planner has \$2 to distribute. First we have the loan management cost as;

$$\kappa(\mu_i(\mathbf{r}^*))^2 = 6.35 \text{ and,}$$
$$\kappa(\mu_j(\mathbf{r}^*))^2 = 6.17.$$

We have $\Phi = \{\phi_1, \phi_2, \phi_3\}$ where $\phi_1 = \{i, j\}$, $\phi_2 = \{i\}$ and $\phi_3 = \{j\}$.

For the strategy ϕ_1, $\gamma_i = \gamma_j = \gamma$. This gives the value as $\gamma = -0.1587$. Strategy ϕ_2 gives $\gamma_i = -0.3149$ while Strategy ϕ_3 gives $\gamma_j = -0.324$.

Strategy 1(ϕ_1): Where $\gamma = -0.1587$.

We have the welfare improvement then as;

$$\begin{aligned}\Delta W^{\gamma}(\mathbf{r}^{\gamma}, \mathcal{A}, a) &= \mathbf{1}^T\mathbf{P}_{\mathcal{A}} \cdot \frac{-\gamma}{(1+\gamma)}, \\ &= 34.751 \cdot \frac{0.1587}{0.8413}, \\ &= 6.56.\end{aligned}$$

Strategy 2(ϕ_2): Where $\gamma_i = -0.3149$.

Let $\eta = \frac{2+a}{4a}$ and $\alpha = \frac{a+1}{a}$. The welfare improvement is;

$$\begin{aligned}\Delta W^{\gamma}(\gamma_i, \mathcal{A}, a) &= \frac{-\gamma_i \alpha}{2\kappa(1+\gamma_i)} \bullet \beta_i(\mathbf{G}_{\mathcal{A}}^T, -a) + \frac{\eta\gamma_i}{1+\gamma_i}, \\ &= \frac{0.7085}{0.0548} \bullet (0.8368) - \frac{0.2755}{0.6851}, \\ &= 10.41.\end{aligned}$$

Strategy 3(ϕ_3): Where $\gamma_j = -0.324$.

The welfare improvement is;

$$\begin{aligned}\Delta W^{\gamma}(\gamma_j, \mathcal{A}, a) &= \frac{-\gamma_j \alpha}{2\kappa(1+\gamma_j)} \bullet \beta_j(\mathbf{G}_{\mathcal{A}}^T, -a) + \frac{\eta\gamma_j}{1+\gamma_j}, \\ &= \frac{0.729}{0.0508} \bullet (0.5515) - \frac{0.2835}{0.6760}, \\ &= 7.4928.\end{aligned}$$

Here, we see that the optimal intervention would be to spend the \$2 on subsidising firm i's loan management cost which in itself, gives a total welfare improvement that supersedes splitting proportionately among both active firms. Also noticeable, is the fact that firm i has a greater externality index $\beta_i(\mathbf{G}_{\mathcal{A}}^T, -a)$ in comparison to firm j which corresponds to our results. On a final note, it is worth pointing out that the sub-strategy combination $C(|\mathcal{A}|, b)$, where $1 < b < |\mathcal{A}|$, strategies are known as *group strategy.* This is even more distinct when the number of active firms exceeds 2 ($|\mathcal{A}| > 2$). Our analysis still implies the planner weighs these strategy and indeed, the optimal could be found within such strategy. However, we have focused primarily on individual firms quality which makes it a suitable target. Group based intervention remain unexplored but relevant.

7 Concluding Remarks

We have shown strategic substituting behaviour of firms arising from firms making an inter-temporal lending rate decision so as to make maximum profit in the face of Loan management cost. Such Loan management cost depends on the level of firms efficiency in managing overall debtors as well as creditors. The outcome of this is a substitute game with mostly a unique equilibrium. Our best replies are very likened to notable works such as Blume, Easley, Kleinberg, Kleinberg, and Tardos (2011), Allouch (2015) as well as Bramoullé et al. (2014) without boundaries and Allouch and King (2018a) with boundaries but with slightly different weight and directional properties. We identify neutrality and welfare improving policies given various types of intervention. The main intuition from out model is that transfers can be made in different forms within a firms property such that the planner improves welfare at little to no additional cost. Lastly, we established that interventions targeted at firms who have a relatively higher degree of network centrality based on weak link to debtors yields the most efficient welfare based outcomes. This is because then, raising such firms lending rate yields lower negative spillover to debtor firms.

This work primarily pays more attention to cost coming from loan management and as such gives intuition towards strategic substitute under the assumption that the firm incurs additional cost on the basis of additional volume of loans. A possible critique of this idea would be that to a significant degree, the number of debtors and creditors are also key drivers of loan management cost as well and we out model fails to capture such part. One reason for ignoring this is that it would mean that firms then make decision as to how many incoming and outgoing link to establish, which goes against our fixed network environment as we assume that such decision are made exogenous to the model, hence the network environment we have . As with regards to decisions on lending rates, given that there are host of other factors that might influence a lending rate charged, then it is easily predicted that other forms of interaction including games of complementarity could arise if the focus is on other factors. Also, because we assume a one-shot decision making, we ignore instances where firms could work to increase administrative efficiency. This in itself could lead to new problems including moral hazard (for example, a personnel might not reveal his/her true efficiency as it might alter remuneration). We believe this would make for a vital extension to the model. Another line of extension would be to factor in a welfare whereby the planner weights firms by order of importance such that payoffs are given weights. This could also shed a more realistic lights to impacts of policies to firms.

A Proofs

A.1 Proof of Lemma 1

Intuitions on this concept is briefly discussed in (Bramoullé et al., 2014). Additionally, it should be noted that because $\mathbf{G}$ is a directed graph, then $(\mathbf{I} + a\mathbf{G})$ being positive definite implies

$$1 + a\, \phi_{min}\left(\frac{\mathbf{G} + \mathbf{G}^T}{2}\right) > 0, \tag{A.1}$$

hence the condition. □

A.2 Proof of Proposition 2

Given (2.8), then for the active set $\mathcal{A}$, we would have for firm $i \in \mathcal{A}$ the following;

$$r_{i\in\mathcal{A}} = \pi_i - a \sum_{j\in\mathcal{N}_i^{in}, j\in\mathcal{A}} \frac{b_{ji}}{b_{-i}} r_j. \tag{A.2}$$

Intuitively, any firm $l \in \mathcal{N} - \mathcal{A}$ would be such that the following holds;

$$r_{l\in\mathcal{N}-\mathcal{A}} = \pi_l - a \sum_{j\in\mathcal{N}_l^{in}, j\in\mathcal{A}} \frac{b_{jl}}{b_{-l}} r_j \leq 0,$$

Which then translates to;

$$a \sum_{j\in\mathcal{N}_l^{in}, j\in\mathcal{A}} \frac{b_{jl}}{b_{-l}} r_j \geq \pi_l. \tag{A.3}$$

Writing (A.2) and (A.3) in vector form for the full set $\mathcal{N}$ completes the proof. □

A.3 Proof of Proposition 3

Holding $\mathcal{N} = \mathcal{A}$, since our Nash equilibrium is $\mathbf{r}(\mathbf{G}, a) = (\mathbf{I} + a\mathbf{G})^{-1} \bullet \boldsymbol{\pi}$, we then solve for $\mathbf{r}(\mathbf{G}, a)$ below as follows;

$$\mathbf{r}(\mathbf{G}, \Omega, a) = \begin{bmatrix} (\mathbf{I} + a\mathbf{G}(\mathbf{CC})) & a\mathbf{G}(\mathbf{CP}) \\ a\mathbf{G}(\mathbf{PC}) & \mathbf{I} \end{bmatrix}^{-1} \bullet \begin{bmatrix} \boldsymbol{\pi}_{Cr} \\ \boldsymbol{\pi}_{Pr} \end{bmatrix} \tag{A.4}$$

Using block matrix inversion concept to solve for $\begin{bmatrix} (\mathbf{I}+a\mathbf{G}(\mathbf{CC})) & a\mathbf{G}(\mathbf{CP}) \\ a\mathbf{G}(\mathbf{PC}) & \mathbf{I} \end{bmatrix}$, assume without loss of generality that $(\mathbf{I}+a\mathbf{G}(\mathbf{CC})) = A$, $a\mathbf{G}(\mathbf{CP}) = F$ while $a\mathbf{G}(\mathbf{PC}) = E$ so that

$$\begin{bmatrix} (\mathbf{I}+a\mathbf{G}(\mathbf{CC})) & a\mathbf{G}(\mathbf{CP}) \\ a\mathbf{G}(\mathbf{PC}) & \mathbf{I} \end{bmatrix} = \begin{bmatrix} A & F \\ E & \mathbf{I} \end{bmatrix}.$$

From the Helmert-Wolf blocking inversion method,[10] we have the following;

$$\begin{bmatrix} A & F \\ E & \mathbf{I} \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1}+A^{-1}F(\mathbf{I}-EA^{-1}F)^{-1}EA^{-1} & -A^{-1}F(\mathbf{I}-EA^{-1}F)^{-1} \\ -(\mathbf{I}-EA^{-1}F)^{-1}EA^{-1} & (\mathbf{I}-EA^{-1}F)^{-1} \end{bmatrix}$$

This is so that we have $\Rightarrow$

$$\begin{aligned} \begin{bmatrix} A & F \\ E & \mathbf{I} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \boldsymbol{\pi}_{Cr} \\ \boldsymbol{\pi}_{Pr} \end{bmatrix} &= \begin{bmatrix} (A^{-1}+A^{-1}F(\mathbf{I}-EA^{-1}F)^{-1}EA^{-1})\,\boldsymbol{\pi}_{Cr} & -\,(A^{-1}F(\mathbf{I}-EA^{-1}F)^{-1})\,\boldsymbol{\pi}_{Pr} \\ -\,((\mathbf{I}-EA^{-1}F)^{-1}EA^{-1})\,\boldsymbol{\pi}_{Cr} & ((\mathbf{I}-EA^{-1}F)^{-1})\,\boldsymbol{\pi}_{Pr} \end{bmatrix} \\ &= \begin{bmatrix} A^{-1}\boldsymbol{\pi}_{Cr} - A^{-1}F(\mathbf{I}-EA^{-1}F)^{-1}(\boldsymbol{\pi}_{Pr} - EA^{-1}\boldsymbol{\pi}_{Cr}) \\ (\mathbf{I}-EA^{-1}F)^{-1}(\boldsymbol{\pi}_{Pr} - EA^{-1}\boldsymbol{\pi}_{Cr}) \end{bmatrix} \end{aligned}$$

We then focus on the first line for which we have the following expression $\Rightarrow$

[10] See Wolf (1978).

$$
\begin{aligned}
A^{-1}\boldsymbol{\pi}_{Cr} - A^{-1}F(\mathbf{I}-EA^{-1}F)^{-1}(\boldsymbol{\pi}_{Pr}-EA^{-1}\boldsymbol{\pi}_{Cr}) &= A^{-1}(\mathbf{I}-EA^{-1}F)^{-1}(\mathbf{I}-EA^{-1}F)\boldsymbol{\pi}_{Cr} \\
&\quad - A^{-1}F(\mathbf{I}-EA^{-1}F)^{-1}(\boldsymbol{\pi}_{Pr}-EA^{-1}\boldsymbol{\pi}_{Cr}) \\
&= A^{-1}(\mathbf{I}-EA^{-1}F)^{-1}\boldsymbol{\pi}_{Cr} \\
&\quad - A^{-1}(\mathbf{I}-EA^{-1}F)^{-1}EA^{-1}F\boldsymbol{\pi}_{Cr} \\
&\quad - A^{-1}F(\mathbf{I}-EA^{-1}F)^{-1}\boldsymbol{\pi}_{Pr} \\
&\quad + A^{-1}F(\mathbf{I}-EA^{-1}F)^{-1}EA^{-1}\boldsymbol{\pi}_{Cr} \\
&= A^{-1}(\mathbf{I}-EA^{-1}F)^{-1}\boldsymbol{\pi}_{Cr} \\
&\quad - A^{-1}F(\mathbf{I}-EA^{-1}F)^{-1}\boldsymbol{\pi}_{Pr} \\
&= A^{-1}(\mathbf{I}-EA^{-1}F)^{-1}(\boldsymbol{\pi}_{Cr}-F\boldsymbol{\pi}_{Pr})
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{r}^*_{Cr}(\mathbf{G},\Omega,a) &= A^{-1}(\mathbf{I}-EA^{-1}F)^{-1}(\boldsymbol{\pi}_{Cr}-F\boldsymbol{\pi}_{Pr}) \\
\mathbf{r}^*_{Pr}(\mathbf{G},\Omega,a) &= (\mathbf{I}-EA^{-1}F)^{-1}(\boldsymbol{\pi}_{Pr}-EA^{-1}\boldsymbol{\pi}_{Cr}).
\end{aligned}
$$

Since $F = a\mathbf{G}(\mathbf{CP})$, $E = a\mathbf{G}(\mathbf{PC})$, and $(\mathbf{I}+a\mathbf{G}(\mathbf{CC})) = A$ we then have the following;

$$
\begin{aligned}
\mathbf{r}^*_{Cr}(\mathbf{G},\Omega,a) &= (\mathbf{I}+a\mathbf{G}(\mathbf{CC}))^{-1}\left(\mathbf{I}-a\mathbf{G}(\mathbf{PC})(\mathbf{I}+a\mathbf{G}(\mathbf{CC}))^{-1}a\mathbf{G}(\mathbf{CP})\right)^{-1}(\boldsymbol{\pi}_{Cr}-a\mathbf{G}(\mathbf{CP})\boldsymbol{\pi}_{Pr}), \\
\mathbf{r}^*_{Pr}(\mathbf{G},\Omega,a) &= \left(\mathbf{I}-a\mathbf{G}(\mathbf{PC})(\mathbf{I}+a\mathbf{G}(\mathbf{CC}))^{-1}a\mathbf{G}(\mathbf{CP})\right)^{-1}(\boldsymbol{\pi}_{Pr}-a\mathbf{G}(\mathbf{PC})(\mathbf{I}+a\mathbf{G}(\mathbf{CC}))^{-1}\boldsymbol{\pi}_{Cr}).
\end{aligned}
$$

Since for a matrix $\mathbf{Z}$, $(\mathbf{I}-\theta\bullet\mathbf{I})\bullet\mathbf{Z} = (1-\theta)\bullet\mathbf{Z}$, we then have the following;

$$
\begin{aligned}
\mathbf{r}^*_{Cr}(\mathbf{G},\Omega,a) &= (\mathbf{I}+a\mathbf{G}(\mathbf{CC}))^{-1}\left(\mathbf{I}-a^2\theta\varrho(\mathbf{I}+a\mathbf{G}(\mathbf{CC}))^{-1}\right)^{-1}(\boldsymbol{\pi}_{Cr}-a\theta\boldsymbol{\pi}_{Pr}), \\
&= \left((\mathbf{I}+a\mathbf{G}(\mathbf{CC}))\left(\mathbf{I}-a^2\theta\varrho(\mathbf{I}+a\mathbf{G}(\mathbf{CC}))^{-1}\right)\right)^{-1}(\boldsymbol{\pi}_{Cr}-a\theta\boldsymbol{\pi}_{Pr}), \\
&= \left(\mathbf{I}+a\mathbf{G}(\mathbf{CC})-a^2\theta\varrho\bullet\mathbf{I}\right)^{-1}(\boldsymbol{\pi}_{Cr}-a\theta\boldsymbol{\pi}_{Pr}), \\
&= \left((1-a^2\theta\varrho)\bullet\mathbf{I}+a\mathbf{G}(\mathbf{CC})\right)^{-1}(\boldsymbol{\pi}_{Cr}-a\theta\boldsymbol{\pi}_{Pr}), \\
&= \left(\mathbf{I}+\frac{a}{1-a^2\theta\varrho}\mathbf{G}(\mathbf{CC})\right)^{-1}\frac{(\boldsymbol{\pi}_{Cr}-a\theta\boldsymbol{\pi}_{Pr})}{1-a^2\theta\varrho}.
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{r}^*_{Pr}(\mathbf{G},\Omega,a) &= \left(\mathbf{I}-a^2\theta\varrho(\mathbf{I}+a\mathbf{G}(\mathbf{CC}))^{-1}\right)^{-1}(\boldsymbol{\pi}_{Pr}-a\varrho(\mathbf{I}+a\mathbf{G}(\mathbf{CC}))^{-1}\boldsymbol{\pi}_{Cr}),\\
&= \left((\mathbf{I}+a\mathbf{G}(\mathbf{CC}))^{-1}\left(\mathbf{I}+a\mathbf{G}(\mathbf{CC})-a^2\theta\varrho\bullet\mathbf{I}\right)\right)^{-1}\boldsymbol{\pi}_{Pr}\\
&- a\varrho\left(\left(\mathbf{I}-a^2\theta\varrho(\mathbf{I}+a\mathbf{G}(\mathbf{CC}))^{-1}\right)(\mathbf{I}+a\mathbf{G}(\mathbf{CC}))\right)^{-1}\boldsymbol{\pi}_{Cr},\\
&= (\mathbf{I}+a\mathbf{G}(\mathbf{CC}))\left(\mathbf{I}+a\mathbf{G}(\mathbf{CC})-a^2\theta\varrho\bullet\mathbf{I}\right)^{-1}\boldsymbol{\pi}_{Pr}\\
&- a\varrho\left(\mathbf{I}+a\mathbf{G}(\mathbf{CC})-a^2\theta\varrho\bullet\mathbf{I}\right)^{-1}\boldsymbol{\pi}_{Cr},\\
&= \left(\mathbf{I}+a\mathbf{G}(\mathbf{CC})-a^2\theta\varrho\bullet\mathbf{I}\right)^{-1}((\mathbf{I}+a\mathbf{G}(\mathbf{CC}))\boldsymbol{\pi}_{Pr}-a\varrho\bullet\boldsymbol{\pi}_{Cr})\\
&= \left((1-a^2\theta\varrho)\bullet\mathbf{I}+a\mathbf{G}(\mathbf{CC})\right)^{-1}((\mathbf{I}+a\mathbf{G}(\mathbf{CC}))\boldsymbol{\pi}_{Pr}-a\varrho\bullet\boldsymbol{\pi}_{Cr})\\
&= \left((1-a^2\theta\varrho)\bullet\mathbf{I}+a\mathbf{G}(\mathbf{CC})\right)^{-1}(a\mathbf{G}(\mathbf{CC})\boldsymbol{\pi}_{Pr}+\boldsymbol{\pi}_{Pr}-a\varrho\bullet\boldsymbol{\pi}_{Cr})\\
&= \left((1-a^2\theta\varrho)\bullet\mathbf{I}+a\mathbf{G}(\mathbf{CC})\right)^{-1}\\
&\left(\left((1-a^2\theta\varrho)\bullet\mathbf{I}+a\mathbf{G}(\mathbf{CC})\right)\boldsymbol{\pi}_{Pr}-(1-a^2\theta\varrho)\bullet\boldsymbol{\pi}_{Pr}+\boldsymbol{\pi}_{Pr}-a\varrho\bullet\boldsymbol{\pi}_{Cr}\right)\\
&= \left((1-a^2\theta\varrho)\bullet\mathbf{I}+a\mathbf{G}(\mathbf{CC})\right)^{-1}(a^2\theta\varrho\bullet\boldsymbol{\pi}_{Pr}-a\varrho\bullet\boldsymbol{\pi}_{Cr})\\
&+ ((1-a^2\theta\varrho)\bullet\mathbf{I}+a\mathbf{G}(\mathbf{CC})^{-1}\left((1-a^2\theta\varrho)\bullet\mathbf{I}+a\mathbf{G}(\mathbf{CC})\right)\boldsymbol{\pi}_{Pr}\\
&= \boldsymbol{\pi}_{Pr}+\left((1-a^2\theta\varrho)\bullet\mathbf{I}+a\mathbf{G}(\mathbf{CC})\right)^{-1}(a\varrho(a\theta\bullet\boldsymbol{\pi}_{Pr}-\boldsymbol{\pi}_{Cr}))\\
&= \boldsymbol{\pi}_{Pr}-a\varrho\bullet\left(\mathbf{I}+\frac{a}{1-a^2\theta\varrho}\mathbf{G}(\mathbf{CC})\right)^{-1}\frac{\boldsymbol{\pi}_{Cr}-a\theta\bullet\boldsymbol{\pi}_{Pr}}{1-a^2\theta\varrho}.
\end{aligned}
$$

□

Proof of Proposition 4

From proposition 3, recall we have the vector of Bonacich centrality grouped in Core and Periphery vector as follows;

$$
\mathbf{r}_{Cr}(\mathbf{G},\Omega,a) = \left(\mathbf{I}+\frac{a}{1-a^2\theta\varrho}\mathbf{G}(\mathbf{CC})\right)^{-1}\frac{(\boldsymbol{\pi}_{Cr}-a\theta\boldsymbol{\pi}_{Pr})}{1-a^2\theta\varrho}, \tag{A.5}
$$

as well as,

$$
\mathbf{r}_{Pr}(\mathbf{G},\Omega,a) = \boldsymbol{\pi}_{Pr}-\left(\mathbf{I}+\frac{a}{1-a^2\theta\varrho}\mathbf{G}(\mathbf{CC})\right)^{-1}\frac{(\boldsymbol{\pi}_{Cr}-a\theta\boldsymbol{\pi}_{Pr})}{1-a^2\theta\varrho} \tag{A.6}
$$

Given the assumption that $\mathbf{G}(\mathbf{CC})\bullet\mathbf{1} = \rho\bullet\mathbf{1}$, then such regularity means that we have the following Bonacich centrality based Nash equilibrium vectors;

$$\begin{aligned}\mathbf{r}_{Cr}(\mathbf{G},\Omega,a) &= \left(1+\frac{a\rho}{1-a^2\theta\varrho}\right)^{-1}\frac{(\boldsymbol{\pi}_{Cr}-a\theta\boldsymbol{\pi}_{Pr})}{1-a^2\theta\varrho},\\ &= \left(\frac{1-a^2\theta\varrho}{1-a^2\theta\varrho+a\rho}\right)\frac{(\boldsymbol{\pi}_{Cr}-a\theta\boldsymbol{\pi}_{Pr})}{1-a^2\theta\varrho},\\ &= (1+a\rho-a^2\theta\varrho)^{-1}(\boldsymbol{\pi}_{Cr}-a\theta\boldsymbol{\pi}_{Pr}).\end{aligned}$$

Then for the Periphery set we have;

$$\mathbf{r}_{Pr}(\mathbf{G},\Omega,a) = \boldsymbol{\pi}_{Pr} - a\varrho(1+a\rho-a^2\theta\varrho)^{-1}(\boldsymbol{\pi}_{Cr}-a\theta\boldsymbol{\pi}_{Pr})$$

□

A.4 Proof of Lemma 2

Recall that $\hat{g}_{ji} = \frac{b_{ji}}{b_i}$.

Assume $\mathcal{N} = \mathcal{A}$. This means we can rewrite (2.4) as follows

$$P_i = b_{-i}r_i - \sum_{j\in\mathcal{N}_i^{in}} b_{ji}r_j - \kappa\bullet(\mu_i)^2 \tag{A.7}$$

Also, from (2.8),

$$r_i = \pi_i - a\sum_{j\in\mathcal{N}_i^{in}}\frac{b_{ji}}{b_{-i}}r_j$$

yielding;

$$b_{-i}\pi_i = b_{-i}r_i + a\sum_{j\in\mathcal{N}_i^{in}} b_{ji}r_j \tag{A.8}$$

Also from (A.8),

$$\sum_{j\in\mathcal{N}_i^{in}} b_{ji}r_j = \frac{b_{-i}\pi_i - b_{-i}r_i}{a} \tag{A.9}$$

then substituting (A.8) and (A.9) in (A.7) yields;

$$P_i = b_{-i}r_i - \frac{b_{-i}\pi_i + b_{-i}r_i}{a} - \kappa\bullet(\mu_i)^2$$

which is also;

$$P_i = b_{-i} r_i \frac{(a+1)}{a} - \frac{b_{-i}\pi_i}{a} - \kappa \bullet (b_{-i}\pi_i)^2$$

Given that we have $\pi_i = \frac{1}{2\kappa b_{-i}}$, we then have our payoff as ;

$$P_{i\in\mathcal{A}} = \frac{b_{-i}(a+1)}{a} r_i - \frac{2+a}{4\kappa a}. \tag{A.10}$$

Let $\alpha = \frac{(a+1)}{a}$ and $\eta = \frac{2+a}{4\kappa a}$, given (4), we have the expression with respect to firm $i \in \mathcal{A}$ Bonacich centrality as;

$$P_{i\in\mathcal{A}} = \alpha b_{-i} \left((\mathbf{I} + a\mathbf{G})^{-1} \bullet \boldsymbol{\pi}_{\mathcal{A}} \right)_i - \eta \tag{A.11}$$

In vector for, this becomes;

$$\mathbf{P}_{\mathcal{A}} = diag(\mathbf{B}) \bullet \left((\mathbf{I} + a\mathbf{G})^{-1} \bullet \boldsymbol{\pi}_{\mathcal{A}} \right) - \boldsymbol{K}$$

such that $\boldsymbol{K} = [\eta]^{\mathcal{A}\times 1}$ and $\mathbf{B} = [\alpha \bullet b_{-i}]^{\mathcal{A}\times 1}$. $\square$

A.5 Proof of Proposition 5

So we have that given $\lambda = (1+\varepsilon)$, we have $P_i^{\varepsilon}(r_i) = \lambda \left(b_{-i} \bullet r_i - \sum_{j\in\mathcal{N}_i^{in}} (\hat{g}_{ji} b_i)\, r_j \right) - \kappa\,(\lambda\mu_i)^2 + \Psi r_i$. If we were to take the differential with respect to r_i; we end up with the best reply as follows;

$$r_i(\lambda) = \frac{\pi_i}{\lambda} - a \sum_j \frac{\lambda b_{ji}}{\lambda b_{-i}} r_j = \frac{\pi_i}{\lambda} - a \sum_j \frac{b_{ji}}{b_{-i}} r_j.$$

This is so that rewriting in vector form, our Nash for active firms is given as;

$$\mathbf{r}(\lambda) = (\mathbf{I} + a\mathbf{G}_{\mathcal{A}})^{-1} \bullet \frac{\boldsymbol{\pi}_{\mathcal{A}}}{\lambda}.$$

We can simply deduce from (5.3) that the vector payoff for active firms is as follows;

$$\mathbf{P}_{\mathcal{A}}^{\lambda} = \lambda diag(\mathbf{B}) \bullet \left((\mathbf{I} + a\mathbf{G}_{\mathcal{A}})^{-1} \bullet \frac{\boldsymbol{\pi}_{\mathcal{A}}}{\lambda} \right) - \boldsymbol{K} = \mathbf{P}_{\mathcal{A}}$$

This is because granting εb_{-i} to each $firm - i \in \mathcal{N}$ yields equation (1) and (6). Hence payoff is homogeneous of degree zero, i.e $\mathbf{P}^{\lambda}_{\mathcal{A}}(\lambda b_{-i}) = \mathbf{P}_{\mathcal{A}}(b_{-i})$. As such, welfare differential

$$W(\mathbf{r}^{*}, \mathcal{A}) - W^{\lambda}(\mathbf{r}^{\lambda}, \mathcal{A}) = \mathbf{1}^{T}(\mathbf{P}_{\mathcal{A}} - \mathbf{P}^{\lambda}_{\mathcal{A}}) = 0.$$

□

A.6 Proof of Theorem 1

Assume the planner decides to change a firm i's total lending by a parameter λ and let us have it that the policy intervention λ_i such that payoffs is written as;

$$P^{\lambda}_{i}(r_i) = \lambda_i b_{-i} \bullet r_i - \sum_{j \in \mathcal{N}^{in}_{i}} (\hat{g}_{ji} b_i)\, r_j - k \left(\lambda_i b_{-i} \bullet r_i + a \sum_{j \in \mathcal{N}^{in}_{i}} (\hat{g}_{ji} b_i)\, r_j \right)^2 \tag{A.12}$$

The addition of λb_i to firm i is strictly conditional on the following;

1. $\mathcal{A}(\lambda) = \mathcal{A}$, and
2. $a \in \left]0, \frac{1}{\left|\frac{\mathbf{G}(\lambda)+\mathbf{G}(\lambda)^T}{2}\right|}\right[$.

The Nash equilibrium for firm i given λb_{-i} is ;

$$r^{\lambda}_{i} = \frac{\pi_i}{\lambda_i} - a \sum_{j \in \mathcal{N}^{in}_{i}, j \in \mathcal{A}} \frac{g_{ji}}{\lambda_i} r^{\lambda}_{j}$$

While the equilibrium for all firm $j | j \in \mathcal{N}^{out}_{i} \cap \mathcal{A}$ is

$$r^{\lambda}_{j} = \pi_j - a \sum_{k \in \left(\mathcal{N}^{in}_{k} - \{i\}\right) \cap \mathcal{A}} g_{kj} r^{\lambda}_{k} - a \lambda_i g_{ij} r^{\lambda}_{i}$$

The vector payoff for active firms is then;

$$\mathbf{P}^{\lambda}_{\mathcal{A}} = diag(\mathbf{B}^{\lambda}) \bullet \left((\mathbf{I} + a\mathbf{G}^{\lambda}_{\mathcal{A}})^{-1} \bullet \boldsymbol{\pi}^{\lambda}_{\mathcal{A}} \right) - \boldsymbol{K}$$

where $\mathbf{B}^{\lambda} = (\alpha\lambda_i b_i, \alpha b_j, \alpha b_k, \ldots)^T$, $\boldsymbol{\pi}^{\lambda} = \left(\lambda_i^{-1}\pi_i, \pi_j, \pi_k \ldots\right)^T$ and lastly,

$$\mathbf{G}_{\mathcal{A}}^{\lambda} = \begin{bmatrix} 0 & \frac{g_{ji}}{\lambda_i} & \ldots & \frac{g_{ni}}{\lambda_i} \\ \lambda_i g_{ij} & \ldots & \ldots & g_{ji} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_i g_{in} & g_{nj} & \ldots & 0 \end{bmatrix}$$

We then show that $diag(\mathbf{B}^{\lambda}) \bullet \left((\mathbf{I} + a\mathbf{G}_{\mathcal{A}}^{\lambda})^{-1} \bullet \boldsymbol{\pi}_{\mathcal{A}}^{\lambda}\right) = diag(\mathbf{B}) \bullet ((\mathbf{I} + a\mathbf{G}_{\mathcal{A}})^{-1} \bullet \boldsymbol{\pi}_{\mathcal{A}})$. First we have that

$$diag(\mathbf{B}^{\lambda}) \bullet \left((\mathbf{I} + a\mathbf{G}_{\mathcal{A}}^{\lambda})^{-1} \bullet \boldsymbol{\pi}_{\mathcal{A}}^{\lambda}\right) = \begin{bmatrix} \alpha\lambda_i b_i & 0 & \ldots & 0 \\ 0 & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \alpha b_n \end{bmatrix} \times \begin{bmatrix} m_{ii} & \frac{m_{ji}}{\lambda_i} & \ldots & \frac{m_{ni}}{\lambda_i} \\ m_{ij} * \lambda_i & \ldots & \ldots & m_{ni} \\ \vdots & \vdots & \vdots & \vdots \\ m_{in} * \lambda_i & m_{nj} & \ldots & m_{nn} \end{bmatrix} \times \begin{bmatrix} \frac{\pi_i}{\lambda_i} \\ \pi_j \\ \vdots \\ \pi_n \end{bmatrix}$$

This is then the same as;

$$diag(\mathbf{B}^{\lambda}) \bullet \left((\mathbf{I} + a\mathbf{G}_{\mathcal{A}}^{\lambda})^{-1} \bullet \boldsymbol{\pi}_{\mathcal{A}}^{\lambda}\right) = \begin{bmatrix} \alpha\lambda_i b_{-i} & 0 & \ldots & 0 \\ 0 & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \alpha b_{-n} \end{bmatrix} \times \begin{bmatrix} \frac{1}{\lambda_i}\left(m_{ii}\pi_i + m_{ji}\pi_j + \ldots + m_{ni}\pi_n\right) \\ m_{ij}\pi_i + \ldots + \ldots + m_{ni}\pi_n \\ \vdots \\ m_{in}\pi_i + m_{nj}\pi_j + \ldots + m_{nn}\pi_n \end{bmatrix}$$

$$= \begin{bmatrix} \alpha b_i \left(m_{ii}\pi_i + m_{ji}\pi_j + \ldots + m_{ni}\pi_n\right) \\ \alpha b_j \left(m_{ij}\pi_i + \ldots + \ldots + m_{ni}\pi_n\right) \\ \vdots \\ \alpha b_n \left(m_{in}\pi_i + m_{nj}\pi_j + \ldots + m_{nn}\pi_n\right) \end{bmatrix}$$

$$= diag(\mathbf{B}) \bullet \left((\mathbf{I} + a\mathbf{G}_{\mathcal{A}})^{-1} \bullet \boldsymbol{\pi}_{\mathcal{A}}\right)$$

Say then we have $\lambda_i \neq \lambda_j \neq \ldots \neq \lambda_n$, we have our Nash equilibrium as;

$$(\mathbf{I}+a\mathbf{G}^{\lambda}_{\mathcal{A}})^{-1}\bullet\boldsymbol{\pi}^{\lambda}_{\mathcal{A}} = \begin{bmatrix} m_{ii} & \frac{m_{ji}*\lambda_j}{\lambda_i} & \ldots & \frac{m_{ni}*\lambda_n}{\lambda_i} \\ \frac{m_{ij}*\lambda_i}{\lambda_j} & \ldots & \ldots & \frac{m_{ni}*\lambda_n}{\lambda_j} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{m_{in}*\lambda_i}{\lambda_n} & \frac{m_{nj}*\lambda_j}{\lambda_n} & \ldots & m_{nn} \end{bmatrix} \times \begin{bmatrix} \frac{\pi_i}{\lambda_i} \\ \frac{\pi_j}{\lambda_j} \\ \vdots \\ \frac{\pi_n}{\lambda_n} \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda_i}\left(m_{ii}\pi_i + m_{ji}\pi_j + \ldots + m_{ni}\pi_n\right) \\ \frac{1}{\lambda_j}\left(m_{ij}\pi_i + \ldots + \ldots + m_{ni}\pi_n\right) \\ \vdots \\ \frac{1}{\lambda_j}\left(m_{in}\pi_i + m_{nj}\pi_j + \ldots + m_{nn}\pi_n\right) \end{bmatrix}$$

Which when multiplied by $diag(\mathbf{B}^{\psi})$ still yields the same expression that

$$diag(\mathbf{B}^{\lambda})\bullet\left((\mathbf{I}+a\mathbf{G}^{\lambda}_{\mathcal{A}})^{-1}\bullet\boldsymbol{\pi}^{\lambda}_{\mathcal{A}}\right) = diag(\mathbf{B})\bullet\left((\mathbf{I}+a\mathbf{G}_{\mathcal{A}})^{-1}\bullet\boldsymbol{\pi}_{\mathcal{A}}\right).$$

□

A.7 Proof of Lemma 4

then best replies are ;

$$r^{\gamma}_i = \frac{\pi_i}{(1+\gamma)} - a\sum_{j\in\mathcal{N}^{in}_i, j\in\mathcal{A}} g_{ji}r^{\gamma}_j$$

While the vector payoff for active firms is then;

$$\mathbf{P}^{\gamma}_{\mathcal{A}} = diag(\mathbf{B})\bullet\left((\mathbf{I}+a\mathbf{G}_{\mathcal{A}})^{-1}\bullet\frac{\boldsymbol{\pi}_{\mathcal{A}}}{(1+\gamma)}\right) - \frac{\boldsymbol{K}}{(1+\gamma)} = \frac{1}{(1+\gamma)}\bullet\mathbf{P}_{\mathcal{A}}$$

as such, welfare differential

$$W^{\gamma}(\mathbf{r}^{\gamma},\mathcal{A}) - W(\mathbf{r}^{*},\mathcal{A}) = \mathbf{1}^T\mathbf{P}_{\mathcal{A}}\bullet\frac{\gamma}{(1+\gamma)}$$

□

Proof of Theorem 2

The best replies for the firm i which is subsidised for is ;

$$r^{\gamma}_i = \frac{\pi_i}{(1+\gamma)} - a\sum_{j\in\mathcal{N}^{in}_i, j\in\mathcal{A}} g_{ji}r^{\gamma}_j$$

While the vector payoff for active firms is then;

$$\mathbf{P}^{\gamma}_{\mathcal{A}} = diag(\mathbf{B}) \bullet \left((\mathbf{I} + a\mathbf{G}_{\mathcal{A}})^{-1} \bullet \boldsymbol{\pi}^{\gamma}_{\mathcal{A}}\right) - \boldsymbol{K}^{\gamma}$$

Where $\boldsymbol{\pi}^{\gamma}_{\mathcal{A}} = \left(\frac{\pi_i}{1+\gamma}, \pi_j, \dots\right)^T$, while $\boldsymbol{K}^{\gamma} = \left(\frac{\eta}{1+\gamma}, \eta, \dots\right)^T$. As such, payoff vector differential;

$$\mathbf{P}^{\gamma}(\mathbf{r}^{\gamma}, \mathcal{A}) - \mathbf{P}(\mathbf{r}^{*}, \mathcal{A}) = diag(\mathbf{B}) \bullet \left((\mathbf{I} + a\mathbf{G}_{\mathcal{A}})^{-1} \bullet (\boldsymbol{\pi}^{\gamma}_{\mathcal{A}} - \boldsymbol{\pi}_{\mathcal{A}})\right) - (\boldsymbol{K}^{\gamma} + \boldsymbol{K}) \tag{A.13}$$

Where $\boldsymbol{\pi}^{\gamma}_{\mathcal{A}} - \boldsymbol{\pi}_{\mathcal{A}} = \left(\frac{\pi_i\gamma}{1+\gamma}, 0, \dots, 0\right)^T$, while $\boldsymbol{K}^{\gamma} - \boldsymbol{K} = \left(\frac{\eta\gamma}{1+\gamma}, 0, \dots, 0\right)^T$ We can then expand (A.13) as such;

$$\begin{aligned}
\mathbf{P}^{\gamma}(\mathbf{r}^{\gamma}, \mathcal{A}) - \mathbf{P}(\mathbf{r}, \mathcal{A}) &= \begin{bmatrix} \alpha b_{-i} & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \alpha b_{-n} \end{bmatrix} \bullet (\mathbf{I} + a\mathbf{G}_{\mathcal{A}})^{-1} \bullet \begin{bmatrix} -\frac{\pi_i\gamma}{1+\gamma} \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} -\frac{\eta\gamma}{1+\gamma} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \alpha b_{-i} & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \alpha b_{-n} \end{bmatrix} \bullet \begin{bmatrix} -m_{ii}\frac{\pi_i\gamma}{1+\gamma} \\ -m_{ij}\frac{\pi_i\gamma}{1+\gamma} \\ \vdots \\ -m_{ik}\frac{\pi_i\gamma}{1+\gamma} \end{bmatrix} - \begin{bmatrix} -\frac{\eta\gamma}{1+\gamma} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -m_{ii}\alpha\frac{b_{-i}\pi_i\gamma}{1+\gamma} \\ -m_{ij}\alpha\frac{b_{-j}\pi_i\gamma}{1+\gamma} \\ \vdots \\ -m_{ik}\alpha\frac{b_{-n}\pi_i\gamma}{1+\gamma} \end{bmatrix} - \begin{bmatrix} -\frac{\eta\gamma}{1+\gamma} \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\end{aligned}$$

This then means that since $\pi_i = \frac{1}{2\kappa b_{-i}}$, we have;

$$\Delta W(\mathbf{r}^{\gamma}, \mathcal{A}|i) = \frac{-\gamma\alpha}{2\kappa(1+\gamma)} \left(m_{ii} + m_{ij}\frac{b_{-j}}{b_{-i}} + \dots + m_{in}\frac{b_{-n}}{b_{-i}}\right) + \frac{\eta\gamma}{1+\gamma} \tag{A.14}$$

This means that if $b_{-i} = b_{-j}\ \forall\ i, j \in \mathcal{A}$, then we have that the equation above becomes;

$$\begin{aligned}\Delta W(\mathbf{r}^{\gamma}, \mathcal{A}|i) &= \frac{-\gamma\alpha}{2\kappa(1+\gamma)}\left(m_{ii} + m_{ij} + \ldots + m_{in}\right) + \frac{\eta\gamma}{1+\gamma} \\ &= \frac{-\gamma\alpha}{2\kappa(1+\gamma)} \bullet \beta_i(\mathbf{G}_{\mathcal{A}}^T, -a) + \frac{\eta\gamma}{1+\gamma} \\ &> 0 \qquad \text{in so far } \gamma < 0.\end{aligned} \tag{A.15}$$

Observe also that $\frac{-\gamma\alpha}{2\kappa(1+\gamma)}$ as well as $\frac{\eta\gamma}{1+\gamma}$ is common to every active firm. This means that the firm i such that $\beta_i(\mathbf{G}_{\mathcal{A}}^T, -a)$ is greatest achieves the highest value of $\Delta W(\mathbf{r}^{\gamma}, \mathcal{A}|i)$.

□

B Supplementary Information: Computation of Nash Equilibrium

B.1 Basic Algorithm

For this part, we use the following boundary limit for actions such that we have

$$\forall \quad i \in \mathcal{N} \quad r_i \in [0, \pi_i] \qquad and \quad \boldsymbol{\pi} \leq \mathbf{1}. \tag{B.1}$$

So then assuming we have the system given as follows;

$$\mathbf{r}_{\mathcal{A}} = (\mathbf{I} + a\mathcal{G}_{\mathcal{A},\mathcal{A}})^{-1} \cdot \boldsymbol{\pi}_{\mathcal{A}} \tag{B.2}$$

$$\mathbf{r}_{\mathcal{N}-\mathcal{A}} = \mathbf{0} \tag{B.3}$$

We then have the clearing condition from (B.2) and (B.3) as follows;

$$\mathbf{r}_{\mathcal{A}} \geq \mathbf{0} \tag{B.4}$$

$$a\mathcal{G}_{\mathcal{N}-\mathcal{A},\mathcal{A}} \cdot \mathbf{r}_{\mathcal{A}} \geq \boldsymbol{\pi}_{\mathcal{N}-\mathcal{A}} \tag{B.5}$$

We then have the sequence as follows;

1. Assume an initial $\mathcal{A}(0) \subset \mathcal{N}$ such that $|\mathbf{I} + a\mathcal{G}_{\mathcal{A},\mathcal{A}}| \neq 0$ (which always holds true $\forall$ $\mathcal{A}(k)$ given boundary conditions as in lemma 1).[11]

2. Then using $\mathcal{A}(0)$ and $\mathcal{N} - \mathcal{A}(0)$, solve for (B.2) keeping (B.3) as defined.

3. Then check if the conditions meet the requirement of both (B.4) and (B.5).

4. If (B.4) and (B.5) are satisfied given $\mathcal{A}(0)$, End.

5. Otherwise, select another subset $\mathcal{A}(1) \subset \mathcal{N}$.

6. repeat step 2 to step 4 until step 4 is satisfied.

7. ***END!!!***

B.2 Pseudo-Code for Computation

In this part we rewrite the algorithm from the previous subsection in a form that is easily adapted into codes. We have the algorithm then written as follows;

Algorithm 1 Nash Equilibrium Lending rate Rate Algorithm

procedure DEFINE PARAMETERS
  $\mathcal{A}(k) \subset \mathcal{N}$, $\mathcal{N} - \mathcal{A}(k) \subset \mathcal{N}$, $\mathcal{N} - \mathcal{A}(k) \cap \mathcal{A}(k) = \{\}$, $\mathcal{N} - \mathcal{A}(k) \cup \mathcal{A}(k) = \mathcal{N}$.
  $max_k = 2^{|\mathcal{N}|} - 1$ (*loop*)
loop:
  if $k = 1 : 1 : max_k$ **then**
    $\mathbf{r}_{\mathcal{A}(k)} = (\mathbf{I} + a\mathcal{G}_{\mathcal{A}(k),\mathcal{A}(k)})^{-1} \cdot \boldsymbol{\pi}_{\mathcal{A}(k)}$.
    $\mathbf{r}_{\mathcal{N}-\mathcal{A}(k)} = \mathbf{0}$.
End If:
  $\mathbf{r}_{\mathcal{A}(k)} \geq \mathbf{0}$ and,
  $a\mathcal{G}_{\mathcal{N}-\mathcal{A}(k),\mathcal{A}(k)} \cdot \mathbf{r}_{\mathcal{A}(k)} \geq \boldsymbol{\pi}_{\mathcal{N}-\mathcal{A}(k)}$.
Else:
  goto *loop*.

[11]Note that $k = 2^{|\mathcal{N}|} - 1$ from the maximum subset combination of $\mathcal{N}$

References

Abrantes-Metz, R. M., Kraten, M., Metz, A. D., & Seow, G. S. (2012). Libor manipulation? *Journal of Banking & Finance*, *36*(1), 136–150.

Acemoglu, D., Ozdaglar, A., & Tahbaz-Salehi, A. (2015). Systemic risk and stability in financial networks. *The american economic review*, *105*(2), 564–608.

Aldasoro, I., & Angeloni, I. (2015). Input–output-based measures of systemic importance. *Quantitative Finance*, *15*(4), 589–606.

Aldasoro, I., Gatti, D. D., & Faia, E. (2017). Bank networks: Contagion, systemic risk and prudential policy. *Journal of Economic Behavior & Organization*, *142*, 164–188.

Allen, F., & Babus, A. (2008). Networks in finance.

Allouch, N. (2015). On the private provision of public goods on networks. *Journal of Economic Theory*, *157*, 527–552.

Allouch, N., & Jalloul, M. (2016). Strategic defaults in financial networks. *Queen Marys University Working Paper series*.

Allouch, N., & King, M. (2018a). *Constrained public goods in networks* (Tech. Rep.). School of Economics Discussion Papers.

Allouch, N., & King, M. (2018b). A network approach to welfare.

Amini, H., Filipović, D., & Minca, A. (2016). Uniqueness of equilibrium in a payment system with liquidation costs. *Operations Research Letters*, *44*(1), 1–5.

Babus, A. (2016). The formation of financial networks. *The RAND Journal of Economics*, *47*(2), 239–272.

Ballester, C., Calvó-Armengol, A., & Zenou, Y. (2006). Who's who in networks. wanted: The key player. *Econometrica*, *74*(5), 1403–1417.

Belhaj, M., Bervoets, S., & Deroïan, F. (2016). Efficient networks in games with local complementarities. *Theoretical Economics*, *11*(1), 357–380.

Belhaj, M., & Deroïan, F. (2019). Group targeting under networked synergies. *Games and Economic Behavior*, *118*, 29–46.

Bergstrom, T., Blume, L., & Varian, H. (1986). On the private provision of public goods. *Journal of public economics*, *29*(1), 25–49.

Bhattacharya, S., Gale, D., Barnett, W., & Singleton, K. (1985). Preference shocks, liquidity, and central bank policy. *Liquidity and crises*, *35*.

Bilkic, N., Gries, T., et al. (2014). *Destructive agents, finance firms, and systemic risk* (Tech. Rep.). Paderborn University, CIE Center for International Economics.

Blume, L., Easley, D., Kleinberg, J., Kleinberg, R., & Tardos, É. (2011). Network formation in the presence of contagious risk. In *Proceedings of the 12th acm conference on electronic commerce* (pp. 1–10).

Bonacich, P. (1987). Power and centrality: A family of measures. *American journal of sociology*, *92*(5), 1170–1182.

Bramoullé, Y., & Kranton, R. (2007). Public goods in networks. *Journal of Economic Theory*, *135*(1), 478–494.

Bramoullé, Y., Kranton, R., & D'amours, M. (2014). Strategic interaction and networks. *American Economic Review*, *104*(3), 898–930.

Caballero, R. J., & Simsek, A. (2013). Fire sales in a model of complexity. *The Journal of Finance*, *68*(6), 2549–2587.

Cabrales, A., Gottardi, P., & Vega-Redondo, F. (2014). Risk-sharing and contagion in networks.

Chiu, J., Eisenschmidt, J., & Monnet, C. (2020). Relationships in the interbank market. *Review of Economic Dynamics*, *35*, 170–191.

Coulter, B., Shapiro, J., & Zimmerman, P. (2017). A mechanism for libor. *Review of Finance*, *22*(2), 491–520.

Demange, G. (2016). Contagion in financial networks: a threat index. *Management Science*, *64*(2), 955–970.

Deng, A., Sato, I., & Wu, Y. (2007). Homomorphisms, representations and characteristic polynomials of digraphs. *Linear algebra and its applications*, *423*(2-3), 386–407.

Duffie, D., & Dworczak, P. (2014). *Robust benchmark design* (Tech. Rep.). National Bureau of Economic Research.

Eaglesham, J. (2013). Bank made huge bet, and profit, on libor. *Wall Street Journal*, *9*.

Eisenberg, L., & Noe, T. H. (2001). Systemic risk in financial systems. *Management Science*, *47*(2), 236–249.

Eisl, A., Jankowitsch, R., & Subrahmanyam, M. G. (2017). The manipulation potential of libor and euribor. *European Financial Management*, *23*(4), 604–647.

Elliott, M., Golub, B., & Jackson, M. O. (2014). Financial networks and contagion. *The American economic review*, *104*(10), 3115–3153.

Elsinger, H., Lehar, A., & Summer, M. (2006). Risk assessment for banking systems. *Management science*, *52*(9), 1301–1314.

Feinstein, Z. (2017). Financial contagion and asset liquidation strategies. *Operations Research Letters*, *45*(2), 109–114.

Galeotti, A., Ghiglinoy, C., & Goyal, S. (2016). Financial linkages, portfolio choice and systemic risk.

Galeotti, A., Golub, B., & Goyal, S. (2020). Targeting interventions in networks. *Econometrica*, *88*(6), 2445–2471.

Gollier, C., Koehl, P.-F., & Rochet, J.-C. (1997). Risk-taking behavior with limited liability and risk aversion. *Journal of Risk and Insurance*, 347–370.

Greenwood, R., Landier, A., & Thesmar, D. (2015). Vulnerable banks. *Journal of Financial Economics*, *115*(3), 471–485.

Kada, O. (2020). Characteristic polynomials and zeta functions of equitably partitioned graphs. *Linear Algebra and its Applications*, *588*, 471–488.

König, M., Tessone, C. J., & Zenou, Y. (2009). A dynamic model of network formation with strategic interactions.

Lux, T., Fricke, D., et al. (2012). Core-periphery structure in the overnight money market: evidence from the e-mid trading platform. *Computational Economics*.

Morris, S. (2000). Contagion. *The Review of Economic Studies*, *67*(1), 57–78.

Morris, S., & Shin, H. S. (2001). Global games: Theory and applications.

Nier, E., Yang, J., Yorulmazer, T., & Alentorn, A. (2007). Network models and financial stability. *Journal of Economic Dynamics and Control*, *31*(6), 2033–2060.

Rogers, L. C., & Veraart, L. A. (2013). Failure and rescue in an interbank network. *Management Science*, *59*(4), 882–898.

Rosen, J. B. (1965). Existence and uniqueness of equilibrium points for concave n-person games. *Econometrica: Journal of the Econometric Society*, 520–534.

Sui, P., Tanna, S., & Zhou, D. (2020). Financial contagion in a core-periphery interbank network. *The European Journal of Finance*, *26*(7-8), 691–710.

Van Lelyveld, I., et al. (2014). Finding the core: Network structure in interbank markets.

Journal of Banking & Finance, *49*, 27–40.

Warr, P. G. (1983). The private provision of a public good is independent of the distribution of income. *Economics letters*, *13*(2-3), 207–211.

Wells, S. J. (2004). Financial interlinkages in the united kingdom's interbank market and the risk of contagion.

Wolf, H. (1978). The helmert block method, its origin and development. In *Proceedings of the second international symposium on problems related to the redefinition of north american geodetic networks* (pp. 319–326).