Dynamic Many-to-One Matching*

Ahmet Altınok†

Job Market Paper

click here for the most recent version

October 31, 2019

Abstract

We study many-to-one matching markets in a dynamic framework with the following features: Matching is irreversible, participants exogenously join the market over time, each agent is restricted by a quota, and agents are perfectly patient. A form of strategic behavior in such markets emerges: The side with many slots can manipulate the subsequent matching market in their favor via earlier matchings. In such a setting, a natural question arises: Can we analyze a dynamic many-to-one matching market as if it were either a static many-to-one or a dynamic one-to-one market? First, we provide sufficient conditions under which the answer is yes. Second, we show that if these conditions are not met, then the early matchings are "inferior" to the subsequent matchings. Lastly, we extend the model to allow agents on one side to endogenously decide when to join the market. Using this extension, we provide a rationale for the small amount of unraveling observed in the US medical residency matching market compared to the US college-admissions system.

*I am grateful to Hector Chade for his support and guidance, and to Amanda Friedenberg, Natalia Kovrijnykh, and Alejandro Manelli for their great advice and valuable comments. I benefited from many discussions with Diana MacDonald, Andreas Kleiner, Gustavo Ventura, Domenico Ferraro, Laura Doval, and Galina Vereshchagina. This work benefited from helpful discussions with participants at seminars at Arizona State University, University of Arizona, International Conference on Game Theory in Stony Brook, and Southwest Economic Theory Meetings at Santa Barbara. All errors are my own.

†WP Carey School of Business, Department of Economics, Arizona State University

1 Introduction

Matching has many-real life applications such as college admissions and labor markets. In many environments, all agents do not arrive to the market at the same time. For example, most upper-tier colleges in the US have two admission periods, early and regular.[1] Likewise, unemployed workers and job openings become available at different periods in many labor markets.[2] Therefore, many matching processes are inherently dynamic, with participants arriving and being matched over time.

Numerous two-sided matching environments are *many-to-one* matching markets —multiple agents on one side can be matched with one agent on the other side. The prominent examples are college admissions, school choice, and entry-level professional labor markets such as medical residency matching.[3]

Researchers have studied dynamic one-to-one matching environments: Doval (2019), Kadam & Kotowski (2018), Kurino (2009), Baccara et al. (2016), and so on. This paper is the first to analyze dynamic many-to-one matching markets with irreversible matches. A dynamic many-to-one matching environment differs from a dynamic one-to-one matching environment in an important way. Intuitively, once a match is formed in a dynamic one-to-one matching environment, all parties concerned put no weight on the subsequent matches that form in later periods. This is not necessarily true in a dynamic many-to-one matching environment, because the agents on the side with many slots might join the subsequent matching market even if they form some matches in earlier periods.

For exposition, we use the language of the college-admissions problem throughout the paper, and consider two periods: colleges arrive at the market in the first period, whereas the students exogenously arrive over time. Each college has a quota that is to be filled over two periods. Matches formed in the first period cannot be broken in the second period. Agents unmatched in the first period join the second-period matching market. Each agent only cares about her "ultimate" match. Thus, no per-period payoff exists. For simplicity, we assume everyone is perfectly patient, although the results extend to a small amount of discounting.

In such a setting, two natural questions arise. First, do situations exits in which one can analyze a *dynamic* many-to-one matching market as if it were a *static* matching market? Second, do situations exist in which one can analyze a dynamic *many-to-one* matching market as if it were

[1] Twenty percent of respondents to NACAC's 2016 Admission Trends Survey offered early decision(ED). Six percent of all applications for Fall 2016 admission to colleges were received through ED. Between Fall 2015 and Fall 2016, colleges reported an average increase of 5% in the number of ED applicants and 6% in ED admits.

[2] A similar trend is observable in finance and economics job markets for new doctorates. See Coles et al. (2010) and Volkov et al. (2016)

[3] Other examples are Federal court clerkships in the US, and job markets for new doctorates in many majors such as economics, finance, and marketing.

a dynamic *one-to-one* matching market? We study dynamic many-to-one matching markets, and as part of the analysis, answer these questions.

The paper has three main results. First, we provide conditions under which a dynamic many-to-one matching market can be simplified to either a static many-to-one or a dynamic one-to-one market. One naturally questions what to expect if these conditions are not met. Thus, as a second result, we provide a property of dynamic many-to-one matching markets that cannot be simplified to either a static many-to-one or a dynamic one-to-one market. Finally, we relax the assumption of exogenous arrivals, and examine the unraveling incentives under *deferred acceptance*, which is a stable matching algorithm in static settings.

Answering the first question is important because if the answer is yes, we can extend many known properties of a static many-to-one matching market to a dynamic setting. Moreover, we can identify situations where dynamics do not play any role.

To motivate the analysis of the second question, note that many-to-one matching markets are generally not equivalent to one-to-one markets. However, for an important class of preferences —responsive preferences—the answer is yes for static settings. An example of this class is colleges that only have preferences that are increasing in students' scores. In static settings with responsive preferences, one can think of a college with n slots as n separate colleges with only one slot. One might wonder whether the same feature holds for the dynamic environment. If it does, one can simply use a dynamic one-to-one matching market as a tool to analyze a dynamic many-to-one matching market, just as in the static settings.

The novel insight in dynamic many-to-one matching markets is that a college might want to fill some of its restricted quota during early admissions to manipulate the outcome of the regular admissions. In particular, colleges might find it profitable to sacrifice some seats and accept some lower-ranked students early to be able to enroll a higher-ranked student later on.[4] We call such behavior "strategic manipulation via commitment."

We show the answer to both questions is yes as long as preferences rule out the incentives for strategic manipulation via commitment.

Initially, we define a new notion of dynamic group stability, which is the extension of Doval (2019) to our environment. In short, a dynamic matching is a contingent matching plan and is dynamically group stable if it is immune to any contemporary blocking coalition at any period. We also provide a suitable notion of dynamic pairwise stability, which is being immune to two types of blocking coalitions: first, no blocking pair exists at any period, and second, no incentive to wait for the second-period matching market exists. We show that dynamic pairwise stability is necessary and sufficient for dynamic (group) stability in dynamic one-to-one matching mar-

[4] Sönmez (1999) shows that no stable matching mechanism exists that is non-manipulable via pre-arranged matches in many-to-one matching markets.

kets. However, it is not sufficient, and more surprisingly, it is not even necessary for dynamic group stability in dynamic many-to-one matching markets with responsive preferences, unlike in static settings. Intuitively, a college may use first-period matchings as a commitment device, and do not form blocking pairs for marginal increases through individual students. Hence, the colleges may not behave as if they were composed of individual agents such as seats, slots, and so on.

To analyze dynamic many-to-one matching markets, we adapted the *related one-to-one matching market* introduced by Roth and Sotomayor (1992) to a dynamic setting. It is a useful technical trick to compute stability in static markets with responsive preferences. This approach is useful for two reasons. First, there is a well-known equivalence between many-to-one and one-to-one matching markets under static settings with responsive preferences (Roth and Sotomayor (1992)). Second, every statically stable matching is a dynamically stable matching outcome in dynamic one-to-one matching markets, if agents are perfectly patient.[5] Hence, linking a dynamic many-to-one market to a dynamic one-to-one market will allow us to find the relation between static and dynamic many-to-one matching markets.

One can analyze a dynamic many-to-one matching market as if it were static many-to-one or dynamic one-to-one, as long as preferences are responsive and do not exhibit simultaneous cycles: A subset of agents from each side exists whose preferences over agents in the other subset are opposite.[6] Indeed, any dynamically group stable matching outcome is statically stable, absent simultaneous cycles. Therefore, dynamics do not play any role in markets with agents whose preferences are sufficiently aligned. Intuitively, colleges use early matchings to commit to swap some students in the second period. They are happy to do such a swap, but students are not, which defines a simultaneous cycle. Hence, such a swap cannot arise in the absence of simultaneous cycles. In particular, matching early with lower-ranked students would not lead a college to match with a higher-ranked student later on. Although restrictive, acyclic preferences, meaning no simultaneous cycle exists, include some commonly observed classes such as homogeneous preferences.

If preferences exhibit simultaneous cycles, a natural question to ask is when a statically group stable matching is dynamically group stable. We define a notion of "average preferences" that captures the idea that a set of mediocre students are preferred to a set of extreme —very good and very bad—students. The notion of average preferences is akin to the notion of mean-preserving spread.[7] We show that every statically group stable matching is dynamically group

[5] See Lemma E.1 in the Appendix. Doval (2019) also reaches the same conclusion with sufficient patience.

[6] Romero-Medina and Triossi (2013) and Doval (2019), among others, introduce a notion of simultaneous cycle in the preferences. The simultaneous cycle in Doval (2019) incorporates intertemporal preferences.

[7] By borrowing a tool from the theory of vector inequalities —majorization— Proposition 5.1 illustrates the connection between these two concepts.

stable so long as colleges have average and responsive preferences. Strategic manipulation of a statically stable matching requires a college to match worse in the first period. The college would do so only if it will match with a student in the second period, who was not achievable in the static environment. Such a matching essentially yields a more spread-out group of students to the college. Therefore, if no college prefers a more spread-out group of students to a less spread-out one, the incentives for manipulating the static market does not arise. Hence, with a careful choice of the matching algorithm, early matchings —unraveling— can potentially be eliminated. One can skip to section 6 for analysis of such a phenomenon.

Accordingly, if a statically group stable matching is not dynamically group stable, a preference cycle exists, and at least one college in the cycle has extreme preferences. In other words, a student exists whom a college in the cycle ranks so high that it is willing to sacrifice some seats to lower-ranked students to be able to get him or her in the subsequent matching market.

We then analyze the cases under which a static market fails to predict a dynamically group stable matching. We show that if a statically group stable matching is not dynamically group stable, the blocking coalition for the statically group stable matching forms a particular first-period matching: colleges admit students for whom these colleges are not achievable in the static setting. Moreover, they admit higher-ranked students in the subsequent matching market. The intuition would be that the matching with "inferior students" in the first period is used as a "commitment device" not to poach students from each other later on. Therefore, competition for higher-ranked students in the subsequent matching market is lower. We see a similar pattern in the US college-admissions system: Students who are admitted early show inferior quality in various measures such as SAT, class rank, and extracurricular records.[8]

The benchmark model assumes arrivals are exogenous. In the last part of the paper, we relax this assumption. This extension is motivated by real-life applications of many-to-one matching markets such as college admissions and medical-residency matching.[9] Before the matching stage, students play a non-cooperative game in which they choose to join the market either in the first period or in the second period. After arrival decisions, the matching stage begins in which a dynamic matching takes place. The equilibrium notion is very natural: A dynamically stable matching takes place for any arrival decision of the students, and the students best-respond to each other, anticipating the dynamically stable matching that will arise. The challenge here is twofold: First, dynamically stable matchings are not necessarily unique; second, dynamic stability leaves substantial freedom for the choice of a contingent matching plan. To simplify the

[8]See Avery et. al. (2004) for a detailed report.

[9]College admissions in the US is inherently a dynamic matching market whereas the medical residency market is a centralized-static one where *National Residency Matching Program* (*NRMP*) assigns the matching. However, both hospitals and the interns are free to form matches before joining *NRMP*. There is evidence on early matchings in that market, even though it is rare.

analysis, we fix the contingent matching plan to two commonly used statically stable matching algorithms: student —and college— proposing deferred acceptance, henceforth SPDA and CPDA, respectively. Initially, we focus on one-to-one matching markets as a benchmark and extend the analysis for many-to-one markets.

When we fix the contingent matching plan to SPDA, if preferences are responsive and either acyclic or average, all equilibria are outcome equivalent to the student-optimal stable matching of the static environment regardless of the arrivals. Therefore, no student has strict incentives to join the market early. This result supports the evidence on the medical residency matching market; that is, minimal unraveling occurs, although early matches can be observed in this market.

Things are different if the contingent matching plan is fixed to CPDA. Arrivals exist in which some colleges can do strictly better than the college-optimal stable matching of the static environment, for two underlying reasons. First, colleges have leverage because they are present in the market from the very beginning. Second, colleges can discipline themselves and not proceed for pairwise blocks in the first period, because dynamic pairwise stability is not necessary for dynamic group stability. Thus, equilibrium arrivals exist under which some colleges achieve a strictly better outcome than the college-optimal stable matching of the static environment by using early matchings. Therefore, we provide an alternative explanation of why the US college system has used some form of early admissions for a century.[10] Thus, we conclude that the choice of a stable matching algorithm might affect the "level" of unraveling.

1.1 Related Literature

This paper contributes mainly to the dynamic matching literature. Several recent studies introduce stability notions for one-to-one matching markets. Among them, Doval (2019) is the closest to our setting. She identifies the trade-off between matching today and waiting for a better option in an environment where matchings cannot be revised. In other studies, matching opportunities are fixed, whereas pairings can be revised over time: Damiano and Lam (2005), Kurino (2009), Kadam & Kotowski (2018), Kotowski (2019), Liu (2018), and Pereyra (2013). Our contribution to this literature is to introduce a suitable group stability notion for many-to-one matching markets where matchings form over time and are irreversible.[11] We identify a form of strategic behavior that arises in dynamic many-to-one matching markets, which cannot

[10] Avery et. al. (2004) state that colleges use early admissions to screen the students for whom they are the first choice, or the financially liable students. We do not underestimate the significance of such incentives, but provide another incentive for early matchings, that is, strategic manipulation via commitment.

[11] Pereyra (2013) is an exception, among others. He studies a many-to-one matching environment in a dynamic framework, inspired by a real-life assignment problem faced by the Mexican Ministry of Public Education. He allows for matchings to be revised over time, which is the major distinction from this paper.

emerge from the previous studies. It is an important observation because these strategic early matchings can identify markets that are susceptible to unraveling.

Our results are complements to Sonmez (1997) and Sonmez (1999), who show in static settings that no stable solution exists that is non-manipulable via capacities or pre-arranged matches, respectively. We formalize such environments by incorporating manipulation incentives and provide conditions under which they do not arise. A literature investigates the manipulation via endowments in the context of exchange economies: Postlewaite (1979), Sertel (1994), and Thomson (2011). We show that colleges might want to sacrifice some seats to lower-ranked students in earlier periods, which is in the spirit of destroying the endowments in exchange economies.

This paper also contributes to the market-design literature on matching markets. Some researchers focus on dynamic matching markets where players leave the market permanently once matched, from the point of view of optimality as opposed to stability: Baccara et al. (2019), Akbarpour et al. (2017), Anderson et al. (2017), Leshno (2017), Ünver (2010), and Dur (2011). They study the welfare implications of various matching algorithms where players optimally trade off the cost of waiting against the arrival of better matching opportunities.[12] This paper shows that SPDA disciplines unravelling incentives more than CPDA does.

The paper is organized as follows. Section 2 presents an example which illustrates a form of strategic behavior in dynamic many-to-one markets. We present the model in the section 3. Section 4 introduces the notion of dynamic group stability, and discusses the features of it. Section 5 presents the equilibrium analysis. Later in Section 6, we introduce endogenous arrival decisions, and conclude with Section 7. In the Appendix, Section C introduces the tools which are used to compute dynamically group stable matchings. Likewise, all omitted proofs and examples are in the Appendix.

2 Illustrative Example

Dynamic many-to-one matching environments entail a strategic incentive that does not arise in either static many-to-one or dynamic one-to-one matching markets: first period matchings can be used as a commitment device to affect the second period matchings. The following example illustrates this phenomenon.

The market has two sides, denoted as $\mathcal{C}$ for colleges and $\mathcal{S}$ for students. The interaction takes place over two periods, $t = 1, 2$, between two colleges and six students; that is, $\mathcal{C} = \{c_1, c_2\}$ and $\mathcal{S} = \{s_1, s_2, s_3, s_4, s_5, s_6\}$. Assume every agent except s_6 arrives in the market at period 1, and s_6

[12] Dur (2011) study the school choice problem in a dynamic environment where some families have two children and their preferences and priority orders for the younger child depend on the assignment of the elder one.

arrives in period 2. For exposition, let $\mathcal{S}_t$ denote the set of students who arrive in the market at period t. Thus, $\mathcal{S}_1 = \{s_1, s_2, s_3, s_4, s_5\}$ and $\mathcal{S}_2 = \{s_6\}$. Let c_1 have quota $q_1 = 3$ and let c_2 have quota $q_2 = 2$; they have two periods over which they can fill their quota. We assume matches of the first period cannot be broken in the second period, and all (first-period) unmatched students and colleges are available in the second-period matching market.

Table 1: Preferences of Students

s_1	s_2	s_3	s_4	s_5	s_6
c_1	c_1	c_1	c_1	c_1	c_2
c_2	c_2	c_2	c_2	c_2	c_1

Table 2: Preferences of Colleges

c_1	s_6	s_1	s_2	s_3	s_4	s_5
c_2	s_1	s_2	s_3	s_6	$\emptyset$	

Table 1 depicts the preferences for students. It indicates that, other than s_6, all students prefer c_1 to c_2. Colleges have preferences over subsets of students. We assume the preferences over subsets of students are determined by preferences over individual students: s_1 is preferred to s_2 if and only if adding s_1 to some subset of students is preferred to adding s_2 to the same subset. This assumption is essentially the one of "responsive preferences." Table 2 depicts the preferences of colleges over individual students. In addition, we assume c_1 strictly prefers $\{s_4, s_5, s_6\}$ to $\{s_1, s_2, s_3\}$. Note this ranking is not implied by responsive preferences.

If this market were static, a stable matching would exist in which c_1 matches with $\{s_1, s_2, s_3\}$, c_2 matches with $\{s_6\}$, and s_4 and s_5 are unmatched. By responsive preferences, any set that c_1 prefers to this match must include s_6, but s_6 prefers its match to c_1. Similarly, by responsive preferences, any set that c_2 prefers to its match must include one of the students s_1, s_2, or s_3, but they all prefer their match to c_2. Hence, it is a stable matching. In fact, we can easily verify stable matching is *unique*. Denote this matching by $m \equiv \big\{(c_1; s_1, s_2, s_3), (c_2; s_6), (\emptyset, s_4), (\emptyset, s_5)\big\}$.

The unique stable matching of the static market (m) is no longer stable in the dynamic market. To see m is not "dynamically stable," note that $\{c_1, s_4, s_5\}$ can together block this matching: c_1 can form a period-1 matching with s_4 and s_5, and enters the second-period matching market with a quota of 1. Then, there is a unique stable matching in the second period, where s_6 is matched with c_1, whereas s_1 and s_2 are matched with c_2. To see that this second-period matching is stable, note that c_1 prefers s_6 to each of the remaining students, and similarly, by responsive preferences, c_2 prefers $\{s_1, s_2\}$ to each —of size two— subset of the remaining students in the second period. In fact, it is easy to verify that this matching is the unique stable matching in the second period. Thus, the ultimate matching would be that c_1 matches with $\{s_4, s_5, s_6\}$ and c_2 matches with $\{s_1, s_2\}$, whereas s_3 remains unmatched.

Two features of the dynamic interaction are worth noting. First, the same conclusion arises for any choice of student arrivals, provided that s_4 and s_5 arrive in period 1. Second, the matching outcome is not a statically stable matching; in the static market, $\{c_1, s_1, s_2\}$ blocks this matching. This block does not happen in the dynamic market, because the only way to enroll s_6 is by committing to match with s_4 and s_5 early on. By doing so, c_1 shows its willingness to not attract s_1 and s_2 in the second period, because the first-period matchings are irreversible. Avery et. al. (2004) emphasizes a similar behavior in college admissions in the US:

> "A college that favors early applicants will draw applicants who would not have chosen that college at the end of the application process. The college gains overall if the new applicants it attracts are good enough to offset the loss of high-quality students who must be denied admission in the regular pool now that the college is favoring early applicants." (p.186)

By making commitments, a college can strategically manipulate second-period matchings. In particular, college c_1 favors students s_4 and s_5; thus, it rejects $\{s_1, s_2, s_3\}$ in the subsequent market because some slots are filled in the first period. Consequently, student s_6 now is attracted in the second period, since his slot in c_2 is filled with students whom c_1 rejected. As Avery et. al. (2004) also states, c_1 does so since it believes the loss from admitting $\{s_4, s_5\}$ will be offset by the admission of s_6. Note that, no room is available for commitments to strategically manipulate the second-period matchings in a dynamic one-to-one market.

3 Model

The model builds on Doval (2019). The market has two sides: colleges and students. They form matches over two periods, $t = 1, 2$. All colleges are around in both periods, whereas students arrive over time.

A dynamic many-to-one matching market $\mathcal{E}^m = (\mathcal{C}, \mathcal{S}_1, \mathcal{S}_2; \boldsymbol{q}; \boldsymbol{\succsim_{\mathcal{C}}}, \boldsymbol{\succsim_{\mathcal{S}}})$ consists of the following: $\mathcal{C}$ is the set of colleges and $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ is the set of students. Students in $\mathcal{S}_t$ arrive in the market at period t. A profile of quotas is $\boldsymbol{q} \equiv (q_c)_{c\in\mathcal{C}}$, where q_c is the quota restriction for $c \in \mathcal{C}$. The preference profile of colleges over the sets of students is denoted by $\boldsymbol{\succsim_{\mathcal{C}}} \equiv (\succsim_c)_{c\in\mathcal{C}}$, where $\succsim_c$ is the preference relation of college c over $2^{\mathcal{S}}$. Analogously, $\boldsymbol{\succsim_{\mathcal{S}}} \equiv (\succsim_s)_{s\in\mathcal{S}}$ denotes the preference profile of students over colleges and being unmatched; that is, $\succsim_s$ is the preference relation of student s over $\mathcal{C} \cup \{\emptyset\}$.

Throughout the paper, we make assumptions on the preferences: Preferences are complete, transitive, and strict. Moreover, we also assume colleges have responsive preferences.

Definition 3.1. *College c has **responsive preferences** if, for each $I \subseteq \mathcal{S}$, the following hold:*

1. *for each $s \in \mathcal{S} \setminus I$, $\quad I \cup \{s\} \succsim_c I$ if and only if $\{s\} \succsim_c \emptyset$.*
2. *for each $s, s' \in \mathcal{S} \setminus I$, $\quad I \cup \{s\} \succsim_c I \cup \{s'\}$ if and only if $\{s\} \succsim_c \{s'\}$.*

For the purpose of stability in static settings, the responsive-preferences assumption allows us to rely only on colleges' preferences over single students as opposed to subsets of students.

A **t-period matching** m_t is a mapping from the set of students to the set of colleges; that is, for each $t = 1, 2$,

$$m_t : \bigcup_{\tau=1}^{t} \mathcal{S}_\tau \to \mathcal{C} \cup \{\emptyset\},$$

where $|m_t^{-1}(\{c\})| \leq q_c$ for each $c \in \mathcal{C}$. Note that a t-period matching respects the quota condition. We assume a match is irreversible. Thus, a given two-period matching m_2 may not be feasible; it won't be feasible if it conflicts with m_1. With this fact in mind, we say a pair (m_1, m_2) is a **feasible matching** if $m_1(s) \neq \emptyset$ implies $m_1(s) = m_2(s)$.

Let M_t be the set of t-period matchings, and let $M_2(m_1)$ be the set of feasible second period matchings given that the first-period matching is m_1. Then, $\mathbf{M} \equiv \bigcup_{m_1 \in M_1} \big(\{m_1\} \times M_2(m_1)\big)$ is the set of all feasible matchings.

Agents' first-period behavior may depend on the second-period matching that they expect to arise. A contingent matching plan gives rise to this expectation:

Definition 3.2. *A* ***contingent matching*** *is a mapping $\mu : M_1 \to M_2$ such that, for each $m_1 \in M_1$, $(m_1, \mu(m_1))$ is feasible.*

Let $\mathcal{M}$ denote the set of all contingent matchings. Preferences of students induce preferences over M_2; we abuse notation and write $m_2 \succsim_s m_2'$ whenever $m_2(s) \succsim_s m_2'(s)$. (Note that $\mu(\cdot) \in M_2$).Preferences over M_2, in turn, induce preferences over first-period matchings and contingent matchings, so that $(m_1, \mu) \succsim_s (m_1', \mu')$ if and only if $\mu(m_1)$ is preferred by s to $\mu'(m_1')$, and analogously for colleges.

4 Dynamic Group Stability

In a static one-to-one matching market, a (pairwise) stable matching is both individually rational and immune to blocking pairs: a pair of agents who prefer each other as opposed to their partners. In such markets, if a matching is immune to blocking pairs, it is immune to blocking coalitions. This implication is not true for many-to-one markets. Thus, Roth and Sotomayor (1990) define a matching to be group stable if it is immune to any blocking coalition. Group stability is equivalent to pairwise stability in the static many-to-one markets with responsive preferences.

This paper focuses on dynamic group stability. To define the concept, we begin by pointing to the matchings that a coalition can implement.

Definition 4.1. *Let $m_t \in M_t$ be an arbitrary t-period matching. A coalition $(C, S) \subseteq \mathcal{C} \times \mathcal{S}$ can* ***implement*** *m'_t if the following conditions hold:*

1. *for each $s \in S$, $m'_t(s) \in C \cup \{\emptyset\}$*
2. *for each $s \notin S$ with $m_t(s) \notin C$, $m'_t(s) = m_t(s)$*
3. *for each $s \notin S$ with $m_t(s) \in C$, $m'_t(s) \in \{m_t(s), \emptyset\}$.*

The first part states that a new match can only form among coalition members. The second part states that the matching of a student "not linked" to the coalition remains unchanged. The third part states that a student who is not in the coalition but is "linked" to the coalition either retains his original match or goes unmatched.[13]

We define the agents that are "available" in the second period. Fix a period-1 matching m_1. Define

$$\mathcal{C}(m_1) = \{c \in \mathcal{C} : |m_1^{-1}(\{c\})| < q_c\} \qquad \text{and} \qquad \mathcal{S}(m_1) = \{s \in \mathcal{S}_1 : m_1(s) = \emptyset\} \cup \mathcal{S}_2.$$

Given first-period matching m_1, the set of colleges with unfilled quotas is denoted by $\mathcal{C}(m_1)$, and the set of unmatched students is denoted by $\mathcal{S}(m_1)$.

Definition 4.2. (***Period 2 block***) *Fix some μ. A nonempty coalition $(C, S) \subseteq \mathcal{C}(m_1) \times \mathcal{S}(m_1)$ blocks (m_1, μ) with m'_2 if* ***(i)*** *(m_1, m'_2) is a feasible matching,* ***(ii)*** *(C, S) can implement m'_2, and* ***(iii)*** *for each $c \in C$, $(m'_2)^{-1}(\{c\}) \succ_c (\mu(m_1))^{-1}(\{c\})$ and for each $s \in S$, $m'_2(s) \succ_s \mu(m_1)(s)$.*

Note, the second period is simply a static matching market where $\mathcal{C}(m_1)$ and $\mathcal{S}(m_1)$ are the only agents who are present. Thus, the period-2 block is the blocking coalition of a static environment according to group stability.

Definition 4.3. (***Period 1 block***) *Fix some (m_1, μ). A nonempty coalition $(C, S) \subseteq \mathcal{C} \times \mathcal{S}_1$ blocks (m_1, μ) with m'_1 if* ***(i)*** *(C, S) can implement m'_1, and* ***(ii)*** *for each $c \in C$, $(\mu((m'_1))^{-1}(\{c\}) \succ_c (\mu(m_1))^{-1}(\{c\})$ and for each $s \in S$, $\mu(m'_1)(s) \succ_s \mu(m_1)(s)$.*

A period-1 and a period-2 block have the same features. It ensures that the coalition consists of agents who are present in the market, that the coalition can implement the alternative matching m'_t, and that everyone in the coalition becomes strictly better off.

Say a period-t blocking coalition exists for (m_1, μ) if some $C \in \mathcal{C}$ and $S \in \mathcal{S}$ block (m_1, μ) with m'_t. Having defined blocking coalitions in a dynamic setting, a pair (m_1, μ) is dynamically group stable if it is immune to first- and second-period blocks.

[13] Because he can retain his original match, our solution concept is group stability as opposed to core. If we were to consider core, the third part of Definition 4.1 would only allow for going unmatched.

Definition 4.4. (***Dynamic Group Stability***) *A pair (m_1, μ) is dynamically group stable if*

1. *no period-1 blocking coalition exists for (m_1, μ), and*

2. *for each $\tilde{m}_1 \in M_1$, no period-2 blocking coalition exists for $(\tilde{m}_1, \mu)$*

We use "dynamic stability" and "dynamic group stability" synonymously. We next discuss properties of dynamic group stability.

4.1 Features of Dynamic Group Stability

First, note that if everyone arrives to the market at period 2, dynamic group stability corresponds to the standard notion of group stability. However, if everyone arrives in period 1, it is not the standard static group stability, because arrivals in period 1 allow colleges to make commitments. Note, commitment has no role in one-to-one matching markets. So, in dynamic one-to-one matching markets, dynamic stability corresponds to standard static stability if everyone arrives at period 1.

Second, the period-2 matching market that arises depends on the first-period matching. The contingent matching specifies both "on-the-path" matching and "off-the-path" matchings. Dynamic group stability requires "static" group stability for both on-the-path and off-the-path matchings. If it required stability only at on-the-path matching, some dynamically group stable matchings would not be reasonable. The following example illustrates the phenomenon.

Example 4.1. *Let there be a college, c, and two students, s_1 and s_2. College c has only one slot to fill; thus, it is a one-to-one market. Student s_1 is in the market at period 1, whereas s_2 arrives in period 2. Assume c prefers s_2 to s_1, and s_1 to going unmatched. Both students prefer matching with c to going unmatched.*

Intuitively, in any stable matching, c should be matched to s_2. Indeed, this is the case for Definition 4.4. We show that if we did not impose off-path stability, this would not be the case.

Consider (m_1, μ) where $m_1 = \{(c, s_1)\}$ and $\mu(\tilde{m}_1) = \tilde{m}_1$ for all $\tilde{m}_1$. It is stable on-the path because no agent is available besides s_2 in the second period. However, it is not stable off-the path. To see this, consider $\tilde{m}_1 = \emptyset$; that is, c waits for s_2. Given the contingent matching plan μ, we have the following $\mu(\emptyset) = \emptyset$. It is not stable, because (c, s_i) is a blocking coalition for $\mu(\emptyset)$, for each $i = 1, 2$.

Third, in a static matching environment, a matching is individually rational if no single agent blocks it. We extend the notion of individual rationality (IR henceforth) to our environment.

Definition 4.5. (***Individual Block***) *A student s **blocks** (m_1, μ) if $\phi \succ_s \mu(m_1)(s)$. A college c **blocks** (m_1, μ) if some matching m_1' and some $S \subseteq (\mu(m_1'))^{-1}(\{c\}) \backslash (m_1')^{-1}(\{c\})$ exist so that*

1. $m_1'(s) \in \{c, \phi\}$ *if* $m_1(s) = c$*,*

2. $m_1'(s) = m_1(s)$ *if* $m_1(s) \neq c$*, and*

3. $S \cup m_1'(\{c\}) \succ_c (\mu(m_1))^{-1}(\{c\})$*.*

We say that (m_1, μ) is **individually rational** if no agent blocks (m_1, μ). First, note that dynamic group stability implies IR. However, a dynamically group stable matching can involve pairs (c, s) that are not IR in the static version of the market. Example 4.2 illustrates such a situation.

Example 4.2. *Consider the example in section 2 but with one change. The preference ranking of* c_1 *is now* $\succsim_{c_1} = s_6 \succ_{c_1} s_1 \succ_{c_1} s_2 \succ_{c_1} s_3 \succ_{c_1} \emptyset$*. As before, also assume* $\{s_4, s_5, s_6\} \succ_{c_1} \{s_1, s_2, s_3\}$*, which is still consistent with responsive preferences.*

Here, s_4 *and* s_5 *are not individually acceptable by* c_1*. However, matching with* $\{s_4, s_5, s_6\}$ *is individually rational for* c_1 *as* $\{s_4, s_5, s_6\} \succ_{c_1} \emptyset$*.*

In fact, $(m_1, m_2) = \{(c_1; s_4, s_5, s_6), (c_2, s_1, s_2)\}$ *is a dynamically group stable matching outcome, despite the fact that* c_1 *would not individually accept* s_4 *or* s_5*.*

4.2 Group Stability vs. Pairwise Stability

We extend the notion of pairwise stability in the static environment to our dynamic setting.

Definition 4.6. (***Pairwise Block***) *Fix some* (m_1, μ)*.*

1. $(c, s) \in \mathcal{C} \times \mathcal{S}_1$ *is a* ***period 1 blocking pair*** *for* (m_1, μ) *if* $c \succ_s \mu(m_1)(s)$ *and there exists* $S \subseteq (\mu(m_1))^{-1}(\{c\})$ *with* $|S \cup \{s\}| \leq q_c$ *and* $S \cup \{s\} \succ_c (\mu(m_1))^{-1}(\{c\})$

2. $(c, s) \in \mathcal{C}(m_1) \times \mathcal{S}(m_1)$ *is a* ***period 2 blocking pair*** *for* (m_1, μ) *if* $c \succ_s \mu(m_1)(s)$ *and there exists* $S \subseteq (\mu(m_1))^{-1}(\{c\}) \cap \mathcal{S}(m_1)$ *with* $|S \cup \{s\}| \leq q_c$ *and* $S \cup \{s\} \succ_c (\mu(m_1))^{-1}(\{c\})$

A matching is **dynamically pairwise stable** if **(i)** it is individually rational, **(ii)** there is no period 1 blocking pair, and **(iii)** for each first-period matching, there is no period 2 blocking pair.

If everyone arrives at period 2, the market becomes static, and dynamic pairwise stability corresponds to the standard notion of pairwise stability. Note the same is not true for period-1 blocking pairs. The reason is that period-1 blocking coalitions of size two allows for the following phenomenon: A pair of agents just wait not to match with each other, but to match with other agents in period 2. By contrast, blocking pairs don't allow pairs of agents to simply wait and get different second-period matches (i.e., not with each other). This phenomenon is illustrated with the following example.

Example 4.3. (***dynamic pairwise vs. group stability in one-to-one markets***) *Let $\mathcal{C} = \{c_1, c_2, c_3, c_4\}$, $\mathcal{S}_1 = \{s_1, s_2\}$, and let $\mathcal{S}_2 = \{s_3, s_4\}$, and $q_c = 1$ for all $c \in \mathcal{C}$. Table 3 depicts the preferences in this environment.*

c_1	c_2	c_3	c_4
s_3	s_3	s_3	s_4
s_1	s_2	s_1	s_3
s_2		s_4	
		s_2	

s_1	s_2	s_3	s_4
c_3	c_3	c_4	c_3
c_1	c_2	c_2	c_4
	c_1	c_3	
		c_1	

Table 3: The Preference Relations

We will show that there exists a dynamically pairwise stable matching that has a period-1 blocking coalition.

Consider (m_1, μ), where $m_1 :\equiv \big\{(c_1, s_1), (c_2, s_2)\big\}$ and $\mu(m_1) :\equiv \big\{(c_1, s_1), (c_2, s_2), (c_3, s_3), (c_4, s_4)\big\}$. Notice there is no period-2 blocking pair given m_1. Let μ assign the following off-the-path matches:

$$m_{1,1} :\equiv \big\{(c_2, s_2)\big\}, \qquad \mu(m_{1,1}) :\equiv \big\{(c_2, s_2), (c_1, s_1), (c_3, s_3), (c_4, s_4)\big\}$$
$$m_{1,2} :\equiv \big\{(c_1, s_1)\big\}, \qquad \mu(m_{1,2}) :\equiv \big\{(c_1, s_1), (c_2, s_2), (c_3, s_4), (c_4, s_3)\big\}$$
$$m_{1,3} :\equiv \big\{\emptyset\big\} \qquad , \qquad \mu(m_{1,3}) :\equiv \big\{(c_1, s_2), (c_2, s_3), (c_3, s_1), (c_4, s_4)\big\}$$

and for all other $\hat{m}_1 \in M_1 \setminus \big\{m_1, m_{1,1}, m_{1,2}, m_{1,3}\big\}$, define $\mu(\hat{m}_1)$, as an arbitrary stable matching over the available agents. We can easily verify that there is no period-2 blocking pair for (m_1, μ).

Note that there is no blocking pair for (m_1, μ), hence it is a dynamically pairwise stable matching. However it is not dynamically (group) stable since $\{c_2, s_1\}$ blocks it at period 1 by waiting for the second period matching market. With this, c_2 matches with s_3 whom he prefers to s_2, and s_1 matches with c_3 whom she prefers to s_2.

With example 4.3 in mind, we have the following:

Proposition 4.1. *A pair (m_1, μ) is dynamically group stable if and only if it is individually rational and*

1. *for each $\tilde{m}_1 \in M_1$, there is no period-2 blocking pair for $(\tilde{m}_1, \mu)$, and*

2. *there is no period-1 blocking coalition for (m_1, μ).*

Note that dynamic group stability requires "stability" in the second period, and the second period is simply a static matching environment. Because pairwise stability is equivalent to group stability in a static setting with responsive preferences, Proposition 4.1 follows.

As example 4.3 illustrates, it does not suffice to focus on period-1 blocking pairs. The underlying reason is that a group of agents might wait to be matched as described by the contingent matching μ, which we call "waiting blocks." It will prove useful to introduce a modified concept of pairwise stability that is immune to a pair waiting —one that rules out a waiting block.

Definition 4.7. (***Modified Dynamic Pairwise Stability***) *A pair* (m_1, μ) *is dynamically modified pairwise stable if it is individually rational and the followings hold:*

1. *for each* $\tilde{m}_1 \in M_1$*, there is no period-2 pairwise block for* $(\tilde{m}_1, \mu)$ *and*
2. *there is no period-1 pairwise block for* (m_1, μ)*, and*
3. *there is no period-1 blocking coalition for* (m_1, μ) *with* m_1' *such that* $m_1'(s) \in \{m_1(s), \emptyset\}$ *for each* $s \in \mathcal{S}_1$*, .*

The third part of Definition 4.7 is an extra requirement, which eliminates waiting blocks. Recall from example 4.3 that (m_1, μ) is not dynamically group stable, because $\{c_2, s_1\}$ blocks it by simply waiting for the second-period matching market. For the same reason, it is not dynamically modified pairwise stable either.

Modified dynamic pairwise stability implies dynamic pairwise stability, but not vice versa. Also, the two are equivalent in a static environment with responsive preferences. Thus, the modified pairwise stability coincides with group stability in static environments with responsive preferences. The following proposition summarizes the relation in dynamic settings.

Proposition 4.2. *Fix some* $\mathcal{E}^m$.

i. If $q_c = 1$ *for all* $c \in \mathcal{C}$*, then* (m_1, μ) *is dynamically modified pairwise stable if and only if it is dynamically (group) stable.*

ii. If $q_c \neq 1$ *for some* $c \in \mathcal{C}$*, then modified dynamic pairwise stability is neither sufficient nor necessary for dynamic group stability.*

The first part of Proposition 4.2 states that modified dynamic pairwise stability is necessary and sufficient for dynamic stability in dynamic one-to-one matching markets. The second part states that neither dynamic group stability nor modified dynamic pairwise stability implies the other, in dynamic many-to-one matching markets.

To show the first part of Proposition 4.2, we initially focus on the second part. Lemma D.1 implies these two concepts may differ in predictions only due to agents' first-period behavior. In particular, colleges might want to match strategically in the first period to change the outcome in the second period for themselves. But such a behavior occurs if the matched agents join the market in the second period as well. Clearly, it is not the case for dynamic one-to-one markets. Therefore, the first part of Proposition 4.2 follows.

The illustrative example on page 7 serves as a counterexample for the second part of Proposition 4.2; see appendix D.2. Intuitively, a college may have incentive to marginally enhance its portfolio by a student. This incentive might lead it to a major loss, as in our example in section 2. However, colleges can choose to control their instincts for marginal increases by committing to strategically match worse in the first period in a dynamic setting. Hence, surprisingly, pairwise stability is not necessary or sufficient for group stability in dynamic markets with responsive preferences.

The conclusion by Proposition 4.2 is very useful because researchers tend to assume that the results regarding stability from one-to-one markets extend to many-to-one markets with responsive preferences. However, we show that such an assumption can lead to unstable predictions unless strategic manipulation we identified here is ruled out.

5 Equilibrium Analysis

Initially, we focus on the instances in which a static version of the market works fine. In other words, we identify the cases under which dynamics do not play a role. Next, we focus on the instances in which dynamics do matter. We begin pointing to the incentives for strategic manipulation via commitment by going back to our motivating example.

5.1 Illustrative Example: Strategic Manipulation via Commitment

We revisit the illustrative example to identify the mechanism behind the blocking coalition for dynamically stable matching obtained in the related one-to-one market. First, notice the following:

Remark 5.1. *Recall that in the* ***static*** *version of the market introduced on page 7,*

$$m = \{(c_1; s_1, s_2, s_3), (c_2, s_6), (\emptyset, s_4), (\emptyset, s_5)\}$$

is the unique stable matching. Note that $\{s_4, s_5, s_6\} \succ_{c_1} \{s_1, s_2, s_3\}$. Although s_4 and s_5 are willing to form a blocking coalition with c_1 in the static setting, s_6 prefers c_2 to c_1. However, in the ***dynamic*** *environment, s_6 is forced to be matched with c_1.*

The underlying reason in the example is that c_2 fills its quota with better students than s_6, namely, $\{s_1, s_2\}$. Notice c_2 is able to match with s_1 and s_2 because c_1 lets them go by filling their place with s_4 and s_5 in the first period. College c_1 does so because s_6 remains to be picked in the second period, whom c_1 values the highest.

Thus, c_1 sacrifices some seats to lower-ranked students in the short run to be able to reach a highly ranked student in the long run. Notice c_1 only engages such a strategic matching in the first period due to the combination of the following reasons:

(i) colleges c_1 and c_2 have reverse preferences over students s_6 and s_i; that is, $s_6 \succ_{c_1} s_i$, and $s_i \succ_{c_2} s_6$, for $i = 1, 2, 3$;

(ii) students s_6 and s_i have reverse preferences over colleges c_2 and c_1; that is, $c_2 \succ_{s_6} c_1$, and $c_1 \succ_{s_i} c_2$, for $i = 1, 2, 3$;

(iii) the capacity constraints are binding for both colleges; and

(iv) student s_6 is very highly valued by college c_1, whereas others are not as distinct.

All these reasons together encourage c_1 to follow such a strategy. We next show that if no such systematic state of affairs exists, then colleges do not have incentives to act strategically. When the manipulation incentives are off the table, dynamic many-to-one matching markets are not fundamentally different from static many-to-one markets.

5.2 When Dynamics Do Not Matter

In this section, we provide conditions under which dynamics do not play any role.

5.2.1 The Role of Simultaneous Cycles

When the preferences are sufficiently aligned, incentives for strategic manipulation do not arise. Recall from the example that colleges c_1 and c_2 wish to swap students s_1 and s_6, but students would not like that. There, $\langle c_1, s_6, c_2, s_1, c_1 \rangle$ is a "simultaneous cycle."[14] This type of preference plays a crucial role in strategic manipulation via commitment. If the preferences do not exhibit such instances, the strategic manipulation is ruled out. We define a class of preferences that precludes such instances.

Definition 5.1. *Fix $\mathcal{E}^m$. The string $\langle c_0, s_0, c_1, s_1, \ldots, c_n, s_n \rangle$ is a simultaneous cycle:*

1. *for each $i = 1, \ldots, n$, $c_i \succsim_{s_{i-1}} c_{i-1} \succsim_{s_{i-1}} \emptyset$ and $s_i \succsim_{c_i} s_{i-1} \succsim_{c_i} \emptyset$ and*

2. *for some $i \geq j + 2$, either $c_i = c_j$ or $s_i = s_j$, and $c_{i+k} = c_{j+k}$ and $s_{i+k} = s_{j+k}$ for $k = 1, 2 \ldots$.*

[14] In fact, s_1 can be replaced by s_2 or s_3 to obtain different simultaneous cycles.

The simultaneous cycle is not a new concept; it is widely used in the matching literature. Indeed, the number of stable matchings in a static environment is directly linked to the simultaneous cycles. We say that the preferences are **acyclic** if they do not exhibit a simultaneous cycle.

Next, we analyze the relation between dynamic and static markets. Fix a dynamic many-to-one matching environment $\mathcal{E}^m$ and the pair (m_1, μ) on $\mathcal{E}^m$. We construct related static many-to-one market" $\mathcal{E}^s$ and corresponding pair $(\tilde{m}_1, \tilde{\mu})$ on $\mathcal{E}^s$. It simply is the corresponding matching market where all the students arrive in the market at the second period. See Appendix C.2 for a formal treatment.

Theorem 5.1. *Fix $\mathcal{E}^m$ such that the preferences are acyclic. Then, (m_1, μ) is dynamically group stable if and only if $(\tilde{m}_1, \tilde{\mu}) \equiv (\emptyset, \tilde{\mu}(\emptyset))$ is group stable.*

Theorem 5.1 states that the set of dynamically group stable matchings of a dynamic many-to-one matching market is equivalent to that of the related static many-to-one market. Note the result holds regardless of the arrivals. Thus, dynamics do not play a role as long as preferences are responsive and acyclic.

Intuitively, inferior first-period matchings are used as a commitment device to not poach students from each other in the subsequent market. For that commitment to arise, colleges would like to swap some students in the second period. They could not do such an exchange in the static environment, because the students who are being swapped are unhappy about it. Thus, if any of those colleges have more slots to fill, that college cannot commit to not attract the swapped student back. But notice such an interaction only takes place if preferences allow for such an exchange that students do not like, which defines a cycle. Hence, the result follows.

In the example on page 7, colleges c_1 and c_2 have reverse preferences for $s \in \{s_1, s_2\}$ and s_6, and vice versa. Therefore, c_1 and c_2 would like to exchange some students so that they both can be better off. It seems possible even in the static version of the market. However, c_1 cannot commit to not attract $\{s_1, s_2\}$ back, after such an exchange, because more slots remain to be filled by c_1. Thus, c_1 fills those slots in the first period with other lower-ranked students so that it shows its commitment to swap $s \in \{s_1, s_2\}$ with s_6. Thus, first-period matchings are used as a commitment device not to poach students later on.

Although restrictive, acyclic preferences are an important class. Homogeneous preferences, for example, are in this class. In a matching environment where preferences are homogeneous -at least for one side- early matches do not change the outcome as compared to the statically group stable matchings.

Unlike in the US, college admissions in Turkey are static, with the Council of Higher Education assigning students to the colleges. It is a centralized system such that students are ranked based on predetermined rules. The first-ranked student is the most desired one for each college, and then the second-ranked one, and so on. Thus, colleges' preferences are homogeneous, unlike the US college admissions, where preferences are not necessarily aligned. Preferences over students are formed mainly based on the information that colleges have about the students. The only relevant information in Turkish college admissions is the SAT score and GPA in high school, which have the same valuation across colleges. However, colleges in the US have various sources of information about the students, and these sources can potentially vary across colleges. For example, the interview process and alumni parents, among others, provide different information to different colleges about students. Thus, cyclic preferences may arise in the US college-admission system. In light of our first result, early admissions makes a difference as compared to static admissions in the US, whereas it does not in Turkey. Thus, the result suggests a rationale for the existence of early admissions in the US college system for more than a century, unlike the static centralized system of college admissions in Turkey.

Note the strategic manipulation via commitment is directly related to the quota restriction. For colleges to not poach students from each other, the number of slots to be filled must be limited. Hence, if the quota restriction is not binding for any college, dynamics do not play any role, and thus the static version of the market works just fine. See Appendix E.4.

5.3 When Dynamics Matter

In this section, we focus on instances where dynamics matter. First, we provide one condition over the preferences under which static version of the market still yields dynamically group stable matchings. However, potentially more dynamically group stable matchings occur than what related static market predicts.

5.3.1 The Role of Extreme Preferences

Although it is a major one, cyclic preference is not the only reason for strategic manipulation via commitment. In this section, we allow for cyclicity and analyze a case under which strategic manipulation via commitment is ruled out. It turns out we can characterize a subset of dynamically group stable matchings with the static version of the market, by categorizing the preferences further.

Note that responsive preferences can refer to multiple preference rankings across sets of students. As an example, let some college c have responsive preferences and rank single students in $\{s_1, s_2, s_3, s_4\}$ in the following way: $s_1 \succ_c s_2 \succ_c s_3 \succ_c s_4$. Now consider the following rankings of sets of students of size 2 by c:

$$\succsim_c^1 \equiv \{s_1, s_2\} \succ_c \{s_1, s_3\} \succ_c \boxed{\{s_1, s_4\} \succ_c \{s_2, s_3\}} \succ_c \{s_2, s_4\} \succ_c \{s_3, s_4\},$$
$$\succsim_c^2 \equiv \{s_1, s_2\} \succ_c \{s_1, s_3\} \succ_c \boxed{\{s_2, s_3\} \succ_c \{s_1, s_4\}} \succ_c \{s_2, s_4\} \succ_c \{s_3, s_4\}.$$

Note the only difference between $\succsim_c^1$ and $\succsim_c^2$ is the ranking of $\{s_1, s_4\}$ and $\{s_2, s_3\}$. Both $\succsim_c^1$ and $\succsim_c^2$ are consistent with the responsive preference.[15] Although how these two are ranked does not affect the set of stable matchings in static settings, it can play a critical role in dynamic settings. Thus, we further define two classes of preferences:

Definition 5.2. *Let $A, B \subset \mathcal{S}$ such that $A \cap B = \emptyset$, $|A| = |B|$ and*

$$\max A \succ_c \max B \succ_c \min B \succ_c \min A.$$

*Then, c has locally average (extreme) preferences at A and B if $B \succ_c A$ ($A \succ_c B$). We say that c has **average** (**extreme**) preferences if c has locally average (extreme) preferences for any such disjoint subsets of students A and B.*[16]

Extreme and average preferences subsume a very intuitive class that is akin to the notion of "mean-preserving spread." First, we borrow a notion from the theory of vector inequalities:

Definition 5.3 (**Majorization**). *Let $a, b \in \mathbb{R}^n$. We say that a majorizes b, denoted by $a \succ b$, if the following holds: $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$, and $\sum_{i=1}^{k} a_{(i)} \leq \sum_{i=1}^{k} b_{(i)}$, for $k = 1, \ldots, n-1$, where $a_{(\cdot)} = (a_{(1)}, \ldots, a_{(n)})$ is the ordered vector such that the components are rearranged in increasing order; that is, $a_{(i)} \leq a_{(j)}$ whenever $i \leq j$.*

Now, let $A, B \subset \mathcal{S}$ such that $A \cap B = \emptyset$, and $|A| = |B| = n$. Let $u_c : \mathcal{S} \to \mathbb{R}$ and define v_c^A and v_c^B in the following way:

$$v_c^A \equiv (u_c(a_1), u_c(a_2), \ldots, u_c(a_n)), \quad \text{and} \quad v_c^B \equiv (u_c(b_1), u_c(b_2), \ldots, u_c(b_n)),$$

where v_c^A and v_c^B are arranged in increasing order, without loss of generality. Notice that $v_c^A, v_c^B \in \mathbb{R}^n$. We say that A **majorizes** B if v_c^A majorizes v_c^B.

Proposition 5.1. *For any $A, B \subset \mathcal{S}$ such that A majorizes B; that is, $v_c^A \succ v_c^B$:*

i. if $c \in \mathcal{C}$ has average preferences, then $B \succ_c A$

[15] In fact, these two are the only rankings that are consistent with responsiveness.
[16] $\max S = s \in S$ such that $s \succ_c \tilde{s}$ for any $\tilde{s} \in S$, analogously $\min S$ follows.

ii. if $c \in \mathcal{C}$ has extreme preferences, then $A \succ_c B$

Proposition 5.1 immediately follows from the Definition 5.2 and Definition 5.3. One can observe that majorization is akin to the notion of "mean-preserving spread." Thus, a college that has extreme preferences prefers a set of students that is a mean-preserving spread of another one.

In our motivating example, college c_1 ranks $\{s_4, s_5, s_6\}$ higher than $\{s_1, s_2, s_3\}$ and thus engages in a strategic manipulation. However, if this ranking were to be reversed and the rest remained same, c_1 would not commit to match with s_4 and s_5 to strategically manipulate the second-period matching. See example A.1 for an illustration. We generalize this insight by categorizing the responsive preferences further as extreme and average preferences.

Proposition 5.2. *Fix $\mathcal{E}^m$ such that colleges have average preferences. Let $\mathcal{E}^s$ be the related static many-to-one matching market. For any matching $(\emptyset, \tilde{\mu})$ such that $\tilde{\mu}(\emptyset)$ is group stable on $\mathcal{E}^s$, a dynamically group stable matching (m_1, μ) on $\mathcal{E}^m$ exists such that $\mu(m_1) = \tilde{\mu}(\emptyset)$.*

Proposition 5.2 states that any statically group stable matching is also an outcome of a dynamically group stable matching. Thus, forgetting about dynamics would not lead to a dynamically unstable outcome.

Intuitively, strategic manipulation arises if there is a college that would like to admit inferior students in the first period at the expense of a relatively better student. Such an interaction yields more spread-out group of students to the college. Therefore, strategic manipulation is ruled out for environments where the preferences are average, and thus the static version works fine.

Note that there could be dynamically stable matchings which are not statically stable. But our point is that the reverse does not happen.[17] Although proposition 5.2 does not fully characterize the dynamically stable matchings in the original market, it has an important implication: If one wishes to shut down early matches via a self-enforcing dynamic matching —a dynamically group stable matching— he can do so as long as colleges have average and responsive preferences.

5.3.2 When Static Market Fails

Next, we focus on instances where the static version of the market does not give dynamically group stable matchings. We begin by pointing out an immediate corollary.

Corollary 5.1. *Fix $\mathcal{E}^m$. If there is no dynamically stable matching (m_1, μ) such that $\mu(m_1) = \eta$ for some "statically" stable matching η over $\mathcal{C} \cup \mathcal{S}$, then the following hold:*

[17] The motivating example in Doval (2019) is an instance of such cases. See also Example 6.3.

i. Preferences exhibit a cycle.

ii. Let O denote a preference cycle. Then, $|\sum_{c \in O} q_c| \leq |\mathcal{S}|$.

iii. For some $c \in O$, c has locally extreme preferences for some subsets of students.

Corollary 5.1 provides some properties in markets where statically group stable matchings are not dynamically group stable. In other words, it identifies necessary conditions for the static version of some dynamic many-to-one matching market to fail in its predictions.

Recall the period-1 blocking coalition to the statically stable matching in our example on page 7. There, the first-period matching forms between c_1 and $\{s_4, s_5\}$. Notice both s_4 and s_5 are ranked below every student whom c_1 can achieve in the static version of the market; that is, s_1, s_2, and s_3. Moreover, s_6, whom c_1 prefers more than everyone else, is picked by c_1 in the subsequent market. This feature is not an artifact of the example. We show that if a period-1 blocking coalition exists for some statically group stable matching, then each college in the blocking coalition forms a period-1 matching with lower-ranked students. This finding is interesting for two reasons. First, it identifies that colleges use lower-ranked students in earlier periods to swap some higher-ranked ones in the subsequent market. Second, it is relevant from the application point of view: The US college system has a similar feature.

First, we note the existence of strategic manipulation crucially depends on the choice of the "statically stable matching algorithm" in the second period. A slightly different version of our example illustrates the idea:

Example 5.1 (Illustrative example with $\mathbf{q_{c_2} = 3}$). *Consider our example on page 7 with one change; that is, $q_{c_3} = 3$. Let the rest remain as on page 7. Here, $m = \{(c_1; s_1, s_2, s_3), (c_2, s_6)\}$ is still the unique statically stable matching. If c_1 forms a first-period matching with $\{s_4, s_5\}$, that is, $m_1 = \{(c_1; s_4, s_5)\}$, then there are two stable matchings in the subsequent matching market $\mathcal{C}(m_1) \cup \mathcal{S}(m_1)$:*

$$\mu_{\mathcal{C}}(m_1) = \{(c_1; s_4, s_5, s_6), (c_2; s_1, s_2, s_3)\}, \quad \textit{and} \quad \mu_{\mathcal{S}}(m_1) = \{(c_1; s_1, s_4, s_5), (c_2; s_2, s_3, s_6)\},$$

which correspond to the outcome of CPDA and SPDA, respectively.

If CPDA is in use in the second period, c_1 would proceed for the period-1 blocking coalition and match with $\{s_4, s_5\}$. However, it would not do so if SPDA is in use. Thus, the choice of the statically stable matching algorithm in the second period may give rise to a period-1 blocking coalition. Note both versions of our example share a common feature: if a period-1 blocking coalition exists, the first-period matching is between c_1 and $\{s_4, s_5\}$. Hence, our result follows:

Proposition 5.3. *Let $\mathcal{E}^m$ be a dynamic many-to-one matching market, and let μ be a statically stable matching algorithm. If $(\emptyset, \mu)$ is blocked by a coalition $C \cup S$ at period 1 with m_1, then for each $c \in C$, the following hold:*

i. $s' \succ_c s$ *for all* $s' \in (\mu(\emptyset))^{-1}(\{c\})$ *and* $s \in m_1^{-1}(\{c\})$.

ii. $s' \succ_c s$ *for all* $s' \in (\mu(m_1))^{-1}(\{c\}) \setminus m_1^{-1}(\{c\})$ *and* $s \in m_1^{-1}(\{c\})$.

Proposition 5.3 states that if a statically stable matching is blocked by a coalition with a first-period matching, then all the colleges in the coalition will match with students in the first period for whom that college was not achievable in the static market. Moreover, colleges in the coalition admit less preferred students in the first period, whereas the more preferred ones get admitted in the second period. It is relevant for an empirical point of view: Early admissions in the US college system show a similar pattern. Avery et. al. (2004), who study the incentives for colleges regarding early admissions, document the following:

> "Early applicants have slightly lower SAT scores and class rank, and slightly less impressive extracurricular records than do regular applicants . On average, these early applicants scored between 0 and 5 points lower on each section of the SAT-1 than regular applicants to the same colleges." (p.141)

Remark 5.2 ($\mathbf{m_1}$ **is part of a dynamically stable matching**). *Consider the example on page 7. Note that $(\emptyset, \mu^S)$ is not dynamically group stable, although $\mu^S(\emptyset)$ is statically group stable. Because $\{c_1, s_4, s_5\}$ blocks $(\emptyset, \mu^S)$ in the first period with $m_1 = \{(c_1, s_4, s_5)\}$. It is easy to verify that (m_1, μ^S) is a dynamically group stable matching. Thus, the first-period matching that the coalition forms to block the statically stable matching is part of a dynamically group stable matching.*

One might wonder whether the case in remark 5.2 is always true. Unfortunately, further blocking coalitions to (m_1, μ) might occur, which already arises through a blocking coalition to statically stable matching $(\emptyset, \mu)$. See example A.3 in the appendix for an illustration.

5.4 Method of Proof

Our goal is to find the dynamically group stable matchings in a dynamic many-to-one matching market. Roth and Sotomayor (1990) introduce a useful technical trick to compute stability in static many-to-one matching markets with responsive preferences, namely, *related one-to-one matching market.* We follow a similar approach by extending the arguments for a dynamic setting. Thus, we construct a *related dynamic one-to-one market*. In words, it is an artificial market in which college slots are treated as individual agents. Then, we carefully map dynamic matchings from the artificial to the original market. See Appendix C for the construction. Our goal

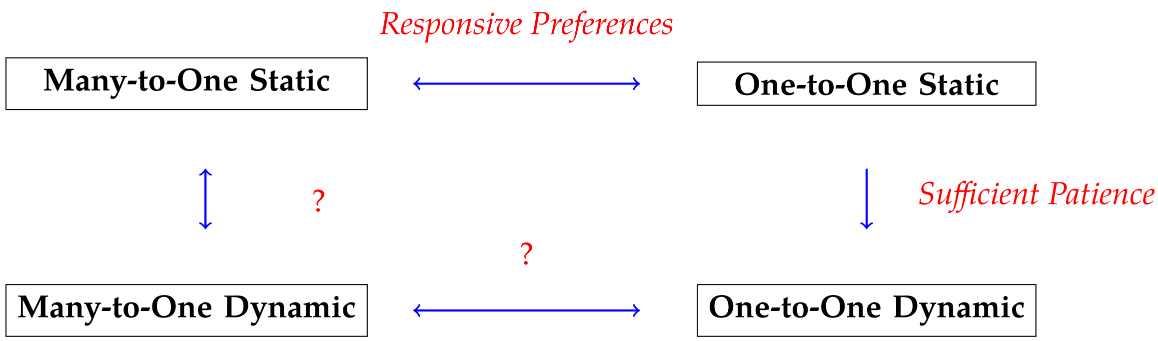

Figure 1: Relation between Many-to-One and One-to-One matching markets

is to identify conditions under which a dynamically group stable matching is obtained via this artificial market. See Appendix C.3 for an illustration of strategic manipulation that does not arise in the related one-to-one market.

Linking dynamic many-to-one market to a dynamic one-to-one market is useful for two reasons. First, any statically stable matching is dynamically stable in dynamic one-to-one matching markets as long as agents are sufficiently patient. Second, the set of stable matchings in a static many-to-one market is equivalent to that of the related static one-to-one market if colleges have responsive preferences. Hence, the diagram in Figure 1 commutes and we are able to link the dynamic many-to-one matching market to a static many-to-one matching market.

6 Endogenous Arrivals

The decision to join the first-period matching market is an endogenous choice of both parties. Although our analysis is based on the fact that students exogenously arrive at the market, the ones who "manipulate" the static matching outcome are strictly better off as opposed to the static outcome. Thus, it might seem like they would endogenously choose to join the first-period matching market anyways. Students' and colleges' incentives to join the first-period matching market is an important channel to study. In this section, we provide a partial answer to such a phenomenon.

Although we would like to understand this phenomenon for many-to-one matching markets, it has not been studied for one-to-one matching markets either. Thus, here we analyze the incentives for one-to-one markets as a benchmark.

We maintain the assumption that colleges are in the market from the beginning of the first period, and add an arrival stage for the students: They simultaneously decide to join the market at either period 1 or period 2. We assume a non-cooperative game in the first stage; that is, each student decides when to enter the market, taking the other students' entry decisions as given.

After the entry stage, the matching stage begins, and the rest of the model is the same as before.

Let $\mathcal{S} \equiv \{s_1, s_2, \cdots, s_n\}$, and $\mathcal{C} \equiv \{c_1, c_2, \cdots, c_m\}$ be the set of students and colleges respectively, where each college c_i has a quota q_{c_i}. The timeline is as follows:

Stage 1: Each $s_i \in \mathcal{S}$, simultaneously, decides whether to join the market in period 1, or wait for the "main" market in period 2.

Stage 2: Given the arrivals, the dynamic matching (m_1, μ) takes place. That is, m_1 is a matching between students who arrive to the market in period 1 and the colleges, and μ is a contingent matching plan that assigns a feasible period-2 matching for every possible first-period matching.

Definition 6.1. *An equilibrium of this game consists of arrival decisions $(a_1, \cdots, a_n)$, $a_i \in \{1, 2\}$, and a dynamic matching (m_1, μ) such that:*

i. given (m_1, μ), each student's action is a best response, and

ii. (m_1, μ) is dynamically group stable for any $(\tilde{a}_1, \cdots, \tilde{a}_n)$.

Clearly, the decision of a student depends solely on the partner he will be matched with through the dynamically stable matching, which need not be unique. Our equilibrium concept requires dynamic stability, which requires the contingent matching plan μ to be stable for all possible first-period matchings. Therefore, we only consider "statically stable" contingent matching plans.

Note that statically stable contingent matching plans can be very general. For example, regardless of all other students' arrivals, we can use SPDA if some particular student arrives in the first period, and CPDA otherwise. One can use many others, because SPDA and CPDA are not the only statically stable matching algorithms. Thus, for a general stable contingent matching plan, one can potentially induce all the students to join the market in the first or second period. To overcome this phenomenon, we fix the contingent matching mechanism μ to two commonly used statically stable matching mechanisms: SPDA and CPDA.

6.1 Student-Proposing Deferred Acceptance: Medical-Residency Market

The National Residency Matching Program (NRMP) in the US has been in use for a very long time. It is a generalized version of intern-proposing deferred acceptance, which assigns the intern-optimal stable matching. Now we consider a dynamic matching market where the intern-optimal stable matching arises in the second period among the agents who are available. As mentioned above, we focus first on the one-to-one case.

As mentioned earlier, we assume the hospitals are always in the market. We want to understand the incentives to join the market early or late from the interns' perspective. We find that as long as all the students are perfectly patient, a unique dynamically stable matching outcome arises, which is equivalent to the intern-optimal stable matching of the static environment. For this section, we only change the notation from colleges and students —$(\mathcal{C}, \mathcal{S})$— to hospitals and interns —$(\mathcal{H}, \mathcal{I})$— respectively.

Let $(\mathcal{H}, \mathcal{I}, \succsim_{\mathcal{H}}, \succsim_{\mathcal{I}})$ be a static one-to-one matching market, and let $m^{\mathcal{H}}$ and $m^{\mathcal{I}}$ denote the intern-optimal and hospital optimal stable matching of the static matching environment, respectively.[18] The following proposition formally states our result:

Proposition 6.1. *Let $(\mathcal{H}, \mathcal{I}_1, \mathcal{I}_2, \succsim_{\mathcal{H}}, \succsim_{\mathcal{I}})$ be an arbitrary dynamic one-to-one matching market where $\mathcal{I}_1 \cup \mathcal{I}_2 = \mathcal{I}$. Assume agents have strict preferences and are perfectly patient. Then, $\mu_{\mathcal{I}}(m_1) = m^{\mathcal{I}}$ for any dynamically stable matching $(m_1, \mu_{\mathcal{I}})$,*

Proposition 6.1 implies that if we assumed the hospital-intern market were a one-to-one matching market, interns would be indifferent between matching early or waiting to join the program, because their partners are the same in all cases. Thus, no student has strict incentives to arrive early. Proposition 6.1 has an obvious implication for the equilibrium pattern:

Corollary 6.1. *Let the contingent matching plan be the intern-proposing deferred acceptance. For any one-to-one matching market with perfect patience and strict preferences, any equilibrium of the endogenous arrivals game exhibits the following properties:*

i. any dynamically stable matching yields intern-optimal stable matching of the static environment for any arrivals,

ii. any arrival decision by the interns is part of an equilibrium.

Because each student is indifferent between arriving to the market in period 1 and period 2 regardless of others' decisions, Corollary 6.1 simply follows. Also, because the unique equilibrium outcome is equivalent to the intern-optimal stable matching of the static environment, we can simply extend the result for some classes of dynamic many-to-one matching markets. The following corollary summarizes:

Corollary 6.2. *Fix a dynamic many-to-one matching market $\mathcal{E}^m$ such that the preferences are transitive, strict, and responsive, and agents are perfectly patient. Then, $\mu_{\mathcal{I}}(m_1) = m^{\mathcal{I}}$ for any dynamically stable matching $(m_1, \mu_{\mathcal{I}})$, if at least one of the followings holds:*

i. preferences do not exhibit any cycle.

ii. $q_h \geq |\mathcal{I}|$ for each $h \in \mathcal{H}$.

[18]Notice that hospitals correspond to colleges, and interns correspond to students from our earlier description. Moreover, note that $m^k \equiv \mu_k(\emptyset)$ for $k = \{\mathcal{H}, \mathcal{I}\}$

iii. each $h \in \mathcal{H}$ has average preferences.

The first and second parts of Corollary 6.2 are consistent with Theorem 5.1 and Proposition E.3 as one should expect. The last part is relevant to the medical-residency matching market, since hospitals would not be expected to act strategically by hiring lower-quality interns at the expense of relatively higher quality interns. Therefore, Corollary 6.2 provides an alternative explanation for why NRMP is considered a successful matching program with little unravelling.

6.2 College-Proposing Deferred Acceptance: College-Admissions Problem

In this section, we consider the college-admissions problem under the assumption that the CPDA algorithm is in use in the second period. We find that colleges can never do worse than the college-optimal stable matching in this case, which is aligned with Proposition 6.1:

Proposition 6.2. *Let $(\mathcal{C}, \mathcal{S}_1, \mathcal{S}_2, \succsim_{\mathcal{C}}, \succsim_{\mathcal{S}})$ be an arbitrary dynamic one-to-one matching market where $\mathcal{S}_1 \cup \mathcal{S}_2 = \mathcal{S}$. Assume agents have strict preferences and are perfectly patient. Then, $\mu_{\mathcal{C}}(m_1) \succsim_{\mathcal{C}} m^{\mathcal{C}}$ for any dynamically stable matching $(m_1, \mu_{\mathcal{C}})$.*

Recall that $m^{\mathcal{C}}$ is the college-optimal stable matching of the static market, and analogously $m^{\mathcal{S}}$. The proof proceeds as follows. First, we show that if no college matches worse in the first period compared to static case, the same holds also for the colleges that match in the second period. Thus, if a college exists that matches worse than statically stable matching, then such a matching exists in the first period as well. However, we show this matching cannot be a dynamically stable matching, because the colleges that match worse in the first period form a period-1 blocking coalition and wait to be matched in the second period.

Unlike the medical-residency matching market, multiple dynamically stable matching outcomes can occur for some arrivals. Example 6.1 illustrates such an instance:

Example 6.1. *Let $\mathcal{C} = \{c_0, c_1, c_2\}$ with $q_c = 1$ for each $c \in \mathcal{C}$, $\mathcal{S}_1 = \{s_0\}$ and $\mathcal{S}_2 = \{s_1, s_2\}$. Table 4 describes the preference relations in this environment:*

c_0	c_1	c_2		s_0	s_1	s_2
s_1	s_1	s_2		c_2	c_2	c_1
s_2	s_2	s_1		c_1	c_1	c_0
s_0	s_0	s_0		c_0	c_0	c_2

Table 4: Preference Relations for example 6.1

When the contingent matching is fixed to CPDA in this market, two dynamically stable matching outcomes arise:

i. $(m_1, \mu_\mathcal{C})$, *where* $m_1 = \{(c_0, s_0)\}$ *and* $\mu_\mathcal{C}(m_1) = \{(c_0, s_0), (c_1, s_1), (c_2, s_2)\}$.

ii. $(\tilde{m}_1, \mu_\mathcal{C})$, *where* $\tilde{m}_1 = \emptyset$ *and* $\mu_\mathcal{C}(\tilde{m}_1) = \{(c_0, s_0), (c_2, s_1), (c_1, s_2)\}$.

As in the example 6.1, some colleges match strictly higher (in some dynamically stable matchings) than the college-optimal stable matching of the static environment when we fix the contingent matching plan to CPDA.

Notice the dynamic stability of $(m_1, \mu_\mathcal{C})$ in example 6.1 relies on the fact that $s_2 \in \mathcal{S}_2$, as (c_0, s_2) forms a blocking pair for $\mu_\mathcal{C}(m_1)$ in the static environment. Thus, if s_2 were in the market in period 1, (c_0, s_2) would block the dynamic matching $(m_1, \mu_\mathcal{C})$. So, the best response of the student s_2 to others' decisions is to join the market at period 1.

Unlike the medical-residency matching market, some students have strict incentives to join the market early in the college-admissions problem. This raises two questions: Is it joining the market in period 1 weakly dominant for students? Do equilibrium arrivals exist under which some colleges match strictly better than in the static case? The following example provides answers to both questions, which is no for the former and yes for the latter:

Example 6.2. *Let* $\mathcal{C} = \{c_0, c_1, c_2, c_3, c_4\}$ *with* $q_c = 1$ *for each* $c \in \mathcal{C}$, $\mathcal{S}_1 = \{s_0, s_4\}$ *and* $\mathcal{S}_2 = \{s_1, s_2, s_3\}$. *Table 5 describes the preference relations in this environment:*

c_0	c_1	c_2	c_3	c_4
s_3	s_2	s_1	s_0	s_3
s_2	s_1	s_2	s_3	s_4
s_0				

s_0	s_4	s_1	s_2	s_3
c_0	c_4	c_1	c_2	c_3
c_3		c_2	c_0	c_4
			c_1	c_0

Table 5: Preference Relations for example 6.2

First, note that $m^\mathcal{C} = \{(c_0, s_0), (c_1, s_1), (c_2, s_2), (c_3, s_3), (c_4, s_4)\} = \mu_\mathcal{C}(\emptyset)$ *is the college-optimal stable matching of the static environment. Consider the dynamically stable matching* $(m_1, \mu_\mathcal{C})$:

$$m_1 = \{(c_0, s_0), (\emptyset, s_4)\} \quad \textit{and} \quad \mu_\mathcal{C}(m_1) = \{(c_0, s_0); (c_1, s_2), (c_2, s_1), (c_3, s_3), (c_4, s_4)\}.$$

Notice that (c_0, s_2) *would be a blocking pair for* $\mu_\mathcal{C}(m_1)$ *in the static environment. It is easy to verify that* $(m_1, \mu_\mathcal{C})$ *is dynamically stable.*

Assume now that s_2 *joins the market in period 1, so that she can block* $(m_1, \mu_\mathcal{C})$ *together with* c_0. *That is,* $\mathcal{S}_1 = \{s_0, s_2, s_4\}$ *and* $\mathcal{S}_2 = \{s_1, s_3\}$. *In this "new" dynamic matching environment, the dynamic matching* $(\tilde{m}_1, \mu_\mathcal{C})$, *defined by*

$$\tilde{m}_1 = \{(\emptyset, s_0), (c_1, s_2), (c_4, s_4)\} \quad \textit{and} \quad \mu_\mathcal{C}(\tilde{m}_1) = \{(c_1, s_2), (c_4, s_4); (c_0, s_3), (c_2, s_1), (c_3, s_0)\},$$

is a dynamically stable matching, because there is no pairwise block and no agents who wishes to wait. Second, s_2 matches with $c_1$under both $(m_1, \mu_\mathcal{C})$ and $(\tilde{m}_1, \mu_\mathcal{C})$. Thus, s_2 does not have incentive to arrive in period 1 to block $(m_1, \mu_\mathcal{C})$, so long as $(\tilde{m}_1, \mu_\mathcal{C})$ arises if she does so.

In the environment described in example 6.2, the arrivals $a_4 = a_0 = 1$ and $a_1 = a_2 = a_3 = 2$, and the dynamic matching $(m_1, \mu_\mathcal{C})$ can be sustained as an equilibrium outcome.

As example 6.2 illustrates, joining the market early is not a weakly dominant strategy. Moreover, equilibrium arrivals exist under which some colleges match strictly higher than in the static environment.

Although students can sometimes block such matchings by arriving early as in example 6.1, the same logic does not extend for many-to-one markets. The underlying reason is that colleges can just discipline themselves and do not go after the blocking pairs. Recall that dynamic pairwise stability is neither necessary nor sufficient for dynamic group stability in dynamic many-to-one matching markets. See the example below for an illustration:

Example 6.3. *Let $\mathcal{C} = \{c_1, c_2\}$ with $q_c = 2$ for each $c \in \mathcal{C}$, and $\mathcal{S} = \{s_0, s_1, s_2, s_3, s_4\}$. Table 6 describes the preference relations in this environment:*

c_1	c_2		s_1	s_2	s_3	s_4
s_1	s_2		c_2	c_1	c_2	c_1
s_2	s_3		c_1	c_2	c_1	c_2
s_3	s_1					
s_4	s_4					

Table 6: Preference Relations for Example 6.3

Note that $m^\mathcal{C} = \{(c_1; s_2, s_4), (c_2; s_1, s_3)\}$ is the college optimal stable matching of the static environment. However, the matching $(m_1, \mu_\mathcal{C})$ where $m_1 = \{(c_1, s_4)\}$, and $\mu_\mathcal{C}(m_1) = \{(c_1; s_1, s_4), (c_2; s_2, s_3)\}$ is dynamically stable as long as s_4 joins the market in the first period, regardless of others' arrivals.

Notice that (c_1, s_2) is a blocking pair for $\mu_\mathcal{C}(m_1)$ in the static environment for example 6.3. It might as well be a blocking pair for $(m_1, \mu_\mathcal{C})$. However, recall from the Proposition 4.2 that dynamic pairwise stability is neither necessary nor sufficient for dynamic group stability. Therefore, c_1 disciplines itself not to pursue a pairwise blocking coalition so that it can reach a strictly better outcome in example 6.3.

As example 6.3 illustrates, colleges can potentially match strictly better by forming some "inferior" first-period matchings, if we fix the contingent matching plan to CPDA.[19] Proposition 6.1

[19]Notice the term "inferior" corresponds to a different set of students in this case from the one Proposition 5.3 states. However, they both emphasize a similar pattern. Sonmez (1999) also identifies such matchings as prearranged matches to manipulate the statically stable matching mechanism.

can shed light on why the medical-residency matching market experiences very little unravelling. Proposition 6.2 supports the evidence on the US college system; that is, some type of early admissions with different labels has always existed, because colleges can be strictly better off via a dynamic matching process.

7 Concluding Remarks

This paper analyzes many-to-one matching markets in a dynamic framework. Applications of such markets include but are not limited to college admissions and various entry-level professional labor markets such as medical-residency matching in the US. In these markets, agents might arrive and form matches sequentially, and breaking a match is considered highly costly for various reasons. By incorporating these key features, we identify a form of strategic behavior that pertains to dynamic many-to-one matching markets. We call such behavior "strategic manipulation via commitment" which exhibits particular early matchings intending to manipulate the outcome of subsequent matching process.

Unless strategic manipulation via commitment is ruled out, dynamic many-to-one matching markets are fundamentally different from static many-to-one and dynamic one-to-one markets. We provide conditions under which a "related" static many-to-one market yields a dynamically stable matching outcome. The conditions are acyclic preferences, average preferences, and high enough quota constraints.

We also analyze the case where agents endogenously choose when to join the market. Our results exhibit similar patterns to the empirical evidence on college admissions in the US and medical residency matching market.

Although they are outside the scope of this paper, the following questions are also worth studying. First, one might wonder whether dynamically group stable matching always exists in environments with responsive preferences and perfect patience. Appendix B.1 provides an example where a dynamically group stable matching does not exist if slight impatience is introduced. Two underlying forces of non-existence are present. First, agents have incentives to form early matches because they are impatient. Second, some slots ought to be left open in the first period for strategic manipulation. These two forces move in opposite directions, and hence can lead to non-existence. Knowing whether the existence with perfect patience can be recovered (because the former incentives disappear) would be interesting. Second, designing a "dynamic" matching algorithm that delivers a dynamically group stable matching, if it exists, would be interesting. Lastly, peer effects in such matching environments are very likely to arise. Thus, incorporating peer effects would be not only interesting but also more realistic for some applications of many-to-one matching.

References

1. Akbarpour M., Li S., and Gharan S. O. "Thickness and information in dynamic matching markets," 2017.

2. Anderson R., I. Ashlagi, D. Gamarnik, and Y. Kanoria (2015): "A dynamic model of barter exchange," in Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms , SIAM, 1925–1933.

3. Avery, Christopher, Andrew Fairbanks and Richard Zeckhauser, The Early Admissions Game: Joining the Elite, Harvard University Press, 2004.

4. Avery, Christopher and Jonathan Levin. 2010. "Early Admissions at Selective Colleges." American Economic Review, 100(5):2125-56

5. Baccara M., Lee S. and Yariv L. (2016) "Optimal dynamic matching".

6. Coles, P., Cawley, J., Levine, P.B., Niederle, M., Roth, A.E., Siegfried, J.J., 2010. "The job market for new economists: a market design perspective." *Journal of Economic Perspectives* 24 (4),187–206

7. Damiano E. and Lam R. (2005). "Stability in dynamic matching markets". *Games and Economic Behavior*, 52(1). pp. 34—53.

8. Doval, L. (2017). "A Theory of Stability in Dynamic Matching Markets".

9. Dur, U. (2011). "Dynamic School Choice Problem".

10. Kadam, S. V. and Kotowski, M. H. (2018) "Multiperiod Matching". *International Economic Review*, 59, 1927–1947

11. Kotowski, M. H. (2019) "A Perfectly Robust Approach to Multiperiod Matching Problems". *HKS Working Paper*

12. Kurino, M. (2009). "Credibility, efficiency and stability: a theory of dynamic matching markets". *Jena economic research papers, JENA*.

13. Lee, Sam-Ho, "Jumping the Curse: Early Contracting with Private Information in University Admissions," *International Economic Review*, Vol. 50, Issue 1, pp. 1-38, February 2009.

14. Leshno, J. (2017): "Dynamic matching in overloaded waiting lists". *Working Paper*

15. Liu, C. (2018) “Stability in Repeated Matching Markets". *Working Paper*

16. Mosler, K. (1994). “Majorization in Economic Disparity Measures," *Linear Algebra and Its Applications*, 199. pp. 91-114

17. Pereyra, J. S. (2013). “A Dynamic School Choice Model". *Games and Economic Behavior*, 80. pp. 100–114

18. Postlewaite, A. (1979). “Manipulation via Endowments", *Review of Economic Studies*, 46. pp. 255—262

19. Pycia, M. (2012). “Stability and preference alignment in matching and coalition formation", *Econometrica*, 80(1). pp. 323—362

20. Romero-Medina A., Triossi M. (2013). “Acyclicity and singleton cores in matching markets" *Economic Letters*, 118 (1), pp. 237-239

21. Roth, A. E. and Sotomayor, M. A. O. (1992). “Two-sided matching: A study in game-theoretic modeling and analysis". *Cambridge University Press*

22. Roth, A. E. and Xing, X. (1994). “Jumping the gun: Imperfections and institutions related to the timing of market transactions". *American Economic Review*, pp. 992—1044

23. Sertel, M. (1994). “Manipulating Lindahl Equilibrium via Endowments". *Economic Letters*, 46. pp. 167—171

24. Sonmez, Tayfun, 1997. “Manipulation via Capacities in Two-Sided Matching Markets?," *Journal of Economic Theory*, 77. pp. 197–204.

25. Sonmez, Tayfun, 1999. “Can Pre-arranged Matches be Avoided in Two-Sided Matching Markets?," *Journal of Economic Theory*, 86. pp. 148–156.

26. Thomson, W. 2011. “Chapter Twenty-One - Fair Allocation Rules," Handbook of Social Choice and Welfare,in: K. J. Arrow & A. K. Sen & K. Suzumura (ed.), *Handbook of Social Choice and Welfare*, edition 1, volume 2, chapter 21, pp. 393–506, Elsevier.

27. Ünver, M. U. (2010): “Dynamic kidney exchange," *The Review of Economic Studies*, 77. pp. 372–414.

28. Volkov, N., Chira, I., Premti, A. (2016). “Who is successful on the finance Ph.D. job market?" *Journal of Corporate Finance*, 37, pp. 109-131.

A Appendix: Omitted Examples

A.1 Examples for section 5.2

Example A.1. *Let $\mathcal{C} = \{c_1, c_2\}$, $\mathcal{S}_1 = \{s_1, s_2, s_3, s_4, s_5\}$ and $\mathcal{S}_2 = \{s_6\}$. Let the preferences of agents be as in tables 1 and 2, and colleges have responsive preferences. Assume that $\{s_1, s_2, s_3\} \succ_{c_1} \{s_4, s_5, s_6\}$ which is still consistent with responsive preferences, and the rest is same as it is described on page 7.*

In this version of the environment, the related dynamic one-to-one market still gives the same outcome; that is, $\{(c_1; s_1, s_2, s_3), (c_2; s_6), (\emptyset, s_4, s_5)\}$. This is in fact the unique statically group stable matching as well. Note here, c_1 would not like to match with $\{s_4, s_5\}$ in the first period to be able to reach s_6 in the second period. It is simply because $\{s_1, s_2, s_3\} \succ_{c_1} \{s_4, s_5, s_6\}$, that is, c_1 ranks an average set of students higher than an extreme one.

Example A.2. *Let $\mathcal{C} = \{c_1, c_2\}$, $\mathcal{S}_1 = \{s_1, s_2, s_3, s_4, s_5\}$ and $\mathcal{S}_2 = \{s_6\}$ be the sets of agents. Also, let the preferences of agents are defined as in tables 1 and 2 and colleges have responsive preferences. Assume that $\{s_6\} \succ_{c_1} \{s_1, s_2, s_3, s_4, s_5\}$ which still is consistent with responsive preferences. Finally, let $q_{c_1} = 5$ and $q_{c_2} = 2$. Notice the set-up is same as in page 7 except college c_1 has five slots to fill. Hence, the number of students are less than the number of slots in the market; that is, $|\mathcal{S}| < |q_{c_1} + q_{c_2}|$.*

*Although it is not obvious, $\{(c_1; s_1, s_2, s_3, s_4, s_5), (c_2; s_6)\}$ is the unique dynamically stable matching outcome of the related one-to-one market. Colleges c_1 and c_2 would still like to swap students $\{s_1, s_2, s_3\}$ and s_6. However, c_1 cannot **commit** to not poach some students $s \in \{s_1, s_2, s_3\}$ in dynamic setting either, as in the static one. Since there are unfilled slots for them in the second period although some slots are filled in the first period. Thus, $\{(c_1; s_1, s_2, s_3, s_4, s_5), (c_2; s_6)\}$ is the unique dynamically group stable matching of the original market as well.*

A.2 Examples for Section 6

Example A.3. *Let $\mathcal{C} = \{c_1, c_2, c_3, c_4\}$, $\mathcal{S}_1 = \{s_4, s_5\}$ and $\mathcal{S}_2 = \{s_1, s_2, s_3, s_6, s_7\}$ where $q_{c_1} = 3$, $q_{c_2} = 2$, and $q_{c_3} = q_{c_4} = 1$. The preferences of colleges over students are **responsive**. Table 7 summarizes the preference relation of each agent in this environment:*

Also assume that $\{s_4, s_5, s_6\} \succ_{c_1} \{s_1, s_2, s_3\}$ as before. Notice there is a unique statically stable matching in this environment:

$$\mu^{\mathcal{C}}(\emptyset) \equiv \{(c_1; s_1, s_2, s_3), (c_2, s_6), (c_3, s_4), (c_4, s_5), (\emptyset, s_7)\} \equiv \mu^{\mathcal{S}}(\emptyset)$$

where μ^i is the -i- offering deferred acceptance algorithm, which delivers the -i- optimal stable matching given a static environment, for $i \in \{\mathcal{S}, \mathcal{C}\}$. Now consider first period matching $m_1 \equiv \big\{(c_1; s_4, s_5)\big\}$.

c_1	c_2	c_3	c_4
s_6	s_1	s_4	s_5
s_1	s_2	s_7	s_4
s_2	s_3	s_5	
s_3	s_6	$\emptyset$	
s_4	$\emptyset$		
s_5			

s_1	s_2	s_3	s_4	s_5	s_6	s_7
c_1	c_1	c_1	c_4	c_3	c_2	c_3
c_2	c_2	c_2	c_1	c_1	c_1	$\emptyset$
			c_3	c_4		

Table 7: Preferences of Colleges and Students - Example A.3

Given m_1, there is a unique stable matching in the second period:

$$\mu^{\mathcal{S}}(m_1) \equiv \{(c_1; s_4, s_5, s_6), (c_2, s_1, s_2), (c_3, s_7), (c_4, \emptyset), (\emptyset, s_3)\} \equiv \mu^{\mathcal{C}}(m_1)$$

Thus the statically stable matching $(\emptyset, \mu^{\mathcal{S}}) \equiv (\emptyset, \mu^{\mathcal{C}})$ is blocked by $\{c_1, s_4, s_5\}$ in the 1^{st} period with $m_1 \equiv \big\{(c_1; s_4, s_5)\big\}$, no matter which stable matching algorithm will be used in the second period. Notice both $(m_1, \mu^{\mathcal{S}})$ and $(m_1, \mu^{\mathcal{C}})$ are blocked by $\{c_5, s_4\}$ at $t = 1$ with $\tilde{m}_1 :\equiv \{(c_1, s_5), (c_5, s_4)\}$.

B Appendix: Non-Existence

Proposition B.1. *There exists a dynamic many-to-one matching environment where every agent is sufficiently patient, and dynamically stable matching does not exist, unlike dynamic one-to-one matching markets.*

Proof of Proposition B.1. The proof is by a counterexample. Example below shows there is no dynamically stable matching in our motivating example with slight revision: every agent is slightly impatient i.e. the ranking over agents remains same but matching early with the same partner is preferred whenever it is possible, and every agent is in the market from the beginning:

Example B.1. *Let $\mathcal{C} = \{c_1, c_2\}$, where $q_{c_1} = 3$ and $q_{c_2} = 2$. $\mathcal{S}_1 = \{s_1, s_2, s_3, s_4, s_5, s_6\}$, and $\mathcal{S}_2 = \emptyset$. The preferences of colleges over students are* ***responsive****. Below is the preference relation of each agent in this environment:*

Also, let $(s_4, s_5, s_6) \succ_{c_1} (s_1, s_2, s_3)$, and $\mu^{-1}(m_1)(c_i) \succ_c \mu'^{-1}(m_1')(c_i)$ whenever $|\mu^{-1}(m_1)(c_i)| > |\mu'^{-1}(m_1')(c_i)|$ for any $c \in \{c_1, c_2\}$, as long as it respect responsive preferences. That is, matching with more students is always strictly preferred by either of the colleges, whenever possible. Moreover, assume that, every agent is ***slightly impatient****. Let $(c, t(c))$ denote matching with c at period $t(c)$. Then $c \succ_s c'$ if and only if $(c, 1) \succ_s (c, 2) \succ_s (c', 1) \succ_s (c', 2)$; which states that the time preference of the students is of secondary importance (Analogously defined for the colleges). Unlike Doval (2017), sufficient patience*

c_1	c_2
s_6	s_1
s_1	s_2
s_2	s_3
s_3	s_6
s_4	$\emptyset$
s_5	

$\{s_i\}_{i=1,\ldots,5}$	s_6
c_1	c_2
c_2	c_1

is not enough for the existence of dynamically stable matching in this environment. Here is a sketch of why there is no dynamically stable matching in this environment:

i. *Notice the unique statically stable matching* ***m*** *assigns $(c_1; s_1, s_2, s_3)$ and $(c_2; s_6)$. This would not be an outcome of a dynamically stable matching as (c_1, c_2, s_4, s_5) blocks in the first period with $\tilde{m}_1 \equiv \{(c_1; s_4, s_5), (c_2, \emptyset)\}$ and subsequent* ***unique*** *stable matching $\mu(\tilde{m}_1) \equiv \{(c_1; s_4, s_5, s_6), (c_2, s_1, s_2)\}$.*

ii. *Any matching with $m_1(s) = \emptyset \neq \mu(m_1)(s)$ for any $s \in \mathcal{S}_1$ is not dynamically stable as $\big(\mu(m_1)(s), s\big)$ blocks by forming the match at period 1. To see this, notice that any outcome $\mu(m_1)$ which has a part that is formed in the second period is blocked with $m_1' \equiv \mu(m_1)$ by $\big\{\mu(m_1)(s), s\big\}_{\{s \in \mathcal{S} | m_1(s) = \emptyset \neq \mu(m_1)(s)\}}$. Thus all the matchings has to form at the first period.*[20]

iii. *With ii. in hand, any (m_1, μ) where $\mu(m_1)$ is not statically stable is also not dynamically stable. To see this, consider any (statically) blocking pair (c, s) to $\mu(m_1)$. The same pair blocks (m_1, μ) in the first period with m_1' where*

$$m_1'(s) = c, \quad m_1'(\dot{s}) = m_1(\dot{s}) = \mu(m_1)(\dot{s}) \quad \forall \dot{s} \neq s.$$

Notice, since all the matchings form at $t = 1$, the continuation matching at this specific first period matching m_1' does not matter for c. Thus any incremental increase in the short run by c cannot be offset via second period matchings. This is why there is not a dynamically stable matching in this environment where everyone is in the market and everyone is slightly impatient.

Thus, for this environment, unlike dynamic one-to-one matching markets, dynamically stable matching ***does not*** *exist.*

It is not an artifact of all the students arriving at $t = 1$. In fact, we do not need s_3 to arrive at $t = 1$, following example shows dynamically stable matching does not exists in this very environment with a little revision; that is, $\mathcal{S}_2 = \{s_3\}$. □

[20] Only true for this specific example. This is always true for dynamic one-to-one matching environment, but not necessarily for many-to-one set up.

C Appendix: Tools to Compute Dynamic Group Stability

We construct artificial static and dynamic markets of a given dynamic many-to-one matching market here.

C.1 Related Dynamic One-to-One Matching Market

Fix a dynamic many-to-one matching environment $\mathcal{E}^m$. Now we define the related dynamic one-to-one matching environment $\mathcal{E}^o = (\hat{\mathcal{X}}, \hat{\mathcal{S}}_1, \hat{\mathcal{S}}_2; \hat{\succsim}_{\mathcal{X}}, \hat{\succsim}_{\mathcal{S}})$ where $\hat{\mathcal{X}} = \left\{x_i^c : c \in C, \quad i = 1, \ldots, q_c\right\}$ is the set of *seat agents* that corresponds to the set of colleges, and $\hat{\mathcal{S}}_t = \mathcal{S}_t$ for each $t = 1, 2$ are the sets of students as before. For each $c \in \mathcal{C}$ and each $i \in \{1, 2, \ldots, q_c\}$, $\hat{\succsim}_{x_i^c} = \succsim_c$ denotes the preferences of seat agents, which is equivalent to the preferences of colleges. Thus $\hat{\succsim}_{\mathcal{X}} = \{\hat{\succsim}_{x_i^c}\}_{x_i^c \in \mathcal{X}}$ denotes the preference profile of seat agents. Similarly, $\hat{\succsim}_{\mathcal{S}} = \{\hat{\succsim}_s\}_{s \in \mathcal{S}}$ denotes the preference profile of students defined by: for each $s \in \mathcal{S}$,

i. $x_i^c \hat{\succsim}_s x_j^{c'}$ if and only if $c \succsim_s c'$, and

ii. $x_i^c \hat{\succsim}_s x_j^c$ if and only if $i \leq j$

Notice, the preference relation of students over seat agents is parallel to that of students over colleges. Part i. states the following: all the seat agent of a higher ranked college are preferred to those of a lower ranked college. Part ii., on the other hand, states that a lower indexed seat agent is always strictly preferred to a higher indexed seat agent from the same college.

Given this mapping, there is a canonical mapping from $(M_1, \mathcal{M})$ in $\mathcal{E}^m$ to $(\hat{M}_1, \hat{\mathcal{M}})$ in $\mathcal{E}^0$:

(i) if $s, s' \in (\mu(m_1))^{-1}(\{c\})$ and $s \succsim_c s'$, then there exists $i < j$ with $\hat{\mu}(\hat{m}_1)(s) = x_i^c$ and $\hat{\mu}(\hat{m}_1)(s') = x_j^c$,

(ii) if $|(\mu(m_1))^{-1}(\{c\})| = k < q_c$, then for each $j > k$, $(\hat{\mu}(\hat{m}_1))^{-1}(x_j^c) = \emptyset$,

(iii) if $m_1(s) = c$, then there exists j such that $\hat{m}_1(s) = x_j^c$.

Note that, one-to-one environment is a special case of many-to-one environment where $q_c = 1$ for all c. Thus all the definitions apply to one-to-one environment as well, so feasibility is preserved. Now we define the related static matching market in the following subsection.

C.2 Related Static Matching Market

Fix a dynamic many-to-one matching environment $\mathcal{E}^m$. Now we define the related static matching environment $\mathcal{E}^s = (\tilde{\mathcal{C}}, \tilde{\mathcal{S}}_1, \tilde{\mathcal{S}}_2; \tilde{\succsim}_{\mathcal{C}}, \tilde{\succsim}_{\mathcal{S}})$ where; $\tilde{\mathcal{C}} = \mathcal{C}$ denotes the set of colleges , and $\tilde{\mathcal{S}}_1 = \emptyset$

and $\tilde{\mathcal{S}}_2 = \mathcal{S}_1 \cup \mathcal{S}_2$ denotes the sets of students. Notice the market is collapsed to be static by taking $\mathcal{S}_1$ as empty set. The rest remains same; that is, $\tilde{\succsim}_{\mathcal{C}}$ denotes the preferences of colleges and $\tilde{\succsim}_{\mathcal{S}}$ denotes the preferences of students.

Given this mapping, there is a canonical mapping from $(M_1, \mathcal{M})$ in $\mathcal{E}^m$ to $(\tilde{M}_1, \tilde{\mathcal{M}})$ in $\mathcal{E}^s$: for each $s \in \mathcal{S}$, $\mu(m_1)(s) = \tilde{\mu}(\tilde{m}_1)(s)$. Notice that in the related static market $\tilde{M}_1 = \{\emptyset\}$. Thus, effectively the mapping is from $(M_1, \mathcal{M})$ to $(\emptyset, \tilde{\mathcal{M}})$, which is why $\mathcal{E}^s$ is the related static matching market. Note also that, one-to-one environment is a special case of many-to-one environment where $q_c = 1$ for all c. Thus analogous mapping applies to dynamic one-to-one matching markets as well.

C.3 Illustrative Example - Related Dynamic One-to-One Market

Given the original dynamic many-to-one matching market on page 7, we construct the related dynamic one-to-one market. Let (x_1, x_2, x_3) denote the seat-agents for c_1, and (y_1, y_2) denote the seat-agents for c_2. Then the related dynamic one-to-one market is the following: $\hat{\mathcal{X}} = \{x_1, x_2, x_3, y_1, y_2\}$, $\hat{\mathcal{S}}_1 = \{s_1, s_2, s_3, s_4, s_5\}$, and $\hat{\mathcal{S}}_2 = \{s_6\}$, with the preference relation as in table 8:

x_1	x_2	x_3	y_1	y_2
s_6	s_6	s_6	s_1	s_1
s_1	s_1	s_1	s_2	s_2
s_2	s_2	s_2	s_3	s_3
s_3	s_3	s_3	s_6	s_6
s_4	s_4	s_4	$\emptyset$	$\emptyset$
s_5	s_5	s_5		

s_1	s_2	s_3	s_4	s_5	s_6
x_1	x_1	x_1	x_1	x_1	y_1
x_2	x_2	x_2	x_2	x_2	y_2
x_3	x_3	x_3	x_3	x_3	x_1
y_1	y_1	y_1	y_1	y_1	x_2
y_2	y_2	y_2	y_2	y_2	x_3

Table 8: Preferences in the Related one-to-one Market

Now, consider $(\hat{m}_1, \hat{\mu})$ in the related one-to-one dynamic matching market where;

$$\hat{m}_1 :\equiv \big\{(x_1, s_1), (x_2, s_2), (x_3, s_3), (y_1, \emptyset), (y_2, \emptyset), (\emptyset, s_4), (\emptyset, s_5)\big\}, \quad \hat{\mu}(\hat{m}_1) = \hat{\mu}^{\mathcal{S}}(\hat{m}_1)$$

where $\hat{\mu}^{\mathcal{S}}$ denotes the student offering deferred acceptance mechanism, given $\hat{m}_1$. Notice that given $\hat{m}_1$, agents who are available in period 2 are $\hat{\mathcal{X}}(\hat{m}_1) = \{y_1, y_2, y_3\}$ and $\hat{\mathcal{S}}(\hat{m}_1) = \{s_4, s_5, s_6\}$, and thus there is a unique stable matching in the second period, $\hat{\mu}(\hat{m}_1)$:

$$\hat{\mu}(\hat{m}_1) :\equiv \big\{(x_1, s_1), (x_2, s_2), (x_3, s_3), (y_1, s_6), (y_2, \emptyset), (\emptyset, s_4), (\emptyset, s_5)\big\}.$$

Notice there is no contemporary pair of agents who prefer each other as opposed to the ones assigned by $\hat{\mu}(\hat{m}_1)$. Moreover, no agent has incentive to wait neither individually nor as a group, given $\hat{\mu}$.[21] Thus $(\hat{m}_1, \hat{\mu})$ is dynamically pairwise stable, and by proposition 4.2, it is dynamically stable, too. Now, the corresponding matching in the original dynamic many-to-one matching market would be (m_1, μ) where;

$$m_1 :\equiv \big\{(c_1; s_1, s_2, s_3), (c_2; \emptyset), (\emptyset, s_4), (\emptyset, s_5)\big\}, \quad \mu(m_1) :\equiv \big\{(c_1; s_1, s_2, s_3), (c_2; s_6), (\emptyset, s_4), (\emptyset, s_5)\big\},$$

and μ is the student offering deferred acceptance mechanism. Notice, (m_1, μ) is **not** dynamically group stable in the original market due to the strategic manipulation in the first period that we identified earlier. That is, $\{c_1, s_4, s_5\}$ is a period-1 block for (m_1, μ) with $\tilde{m}_1 \equiv \big\{(c_1; s_4, s_5)\big\}$. To see this, notice there is a unique stable matching in the second period, given $\tilde{m}_1$; that is, $\mu(\tilde{m}_1) :\equiv \big\{(c_1; s_4, s_5, s_6), (c_2; s_1, s_2)\big\}$.

Since c_1 strictly prefers $\{s_4, s_5, s_6\}$ to $\{s_1, s_2, s_3\}$, and s_4 as well as s_5 strictly prefer c_1 to being unmatched, (m_1, μ) is not dynamically group stable. Thus, related dynamic one-to-one does not predict a group stable outcome of the original many-to-one market for this specific example.

Remark C.1. *The blocking coalition requires two seats in c_1; namely $\{x_2, x_3\}$, to be matched with worse students; namely $\{s_4, s_5\}$, as opposed to what related one-to-one suggests; that is $\{s_2, s_3\}$. In other words, c_1 is willing to be a part of such a coalition even though it loses in some seats, because there is a larger gain for itself via another seat.*

Such a blocking coalition would not arise in related dynamic one-to-one market as $\{x_2, x_3\}$ become strictly worse off.

D Appendix: Some Intermediate Results

D.1 Useful Lemmas

Lemma D.1. *Fix some $\mathcal{E}^m$ where preferences are strict, transitive and responsive. There exists a period-2 blocking coalition for (m_1, μ) if and only if there exists a period-2 blocking pair for (m_1, μ).*

Proof of Lemma D.1. Note, if there is a pair that blocks (m_1, μ) with m_2' then there exists a coalition that does the same. We show the converse. For that, suppose the coalition (C, S) blocks (m_1, μ) with m_2'. Then either ($C = \emptyset$ or $S = \emptyset$), or ($C \neq \emptyset$ and $S \neq \emptyset$). First suppose that $C = \emptyset$. Then by definition $m_2'(s) = \emptyset \succ_s \mu(m_1)(s)$ for $s \in S$. Hence, $(\mu(m_1)(s), s)$ is an individually irrational

[21] Even though there are many other dynamically stable matchings in the related one-to-one market, they are all outcome equivalent to $\hat{\mu}(\hat{m}_1)$.

pair and $(\emptyset, s)$ is a period 2 blocking pair for (m_1, μ). A similar argument follows for the case: $C = \emptyset$.

Now assume ($C \neq \emptyset$ and $S \neq \emptyset$). Fix some $c \in C$. We will show two facts. First $\mu(m_1)^{-1}(\{c\}) \setminus (m_2')^{-1}(\{c\}) \subseteq \mathcal{S}(m_1)$ and $(m_2')^{-1}(\{c\}) \setminus \mu(m_1)^{-1}(\{c\}) \subseteq \mathcal{S}(m_1)$. Second, there exists $s' \in \mu(m_1)^{-1}(\{c\}) \setminus (m_2')^{-1}(\{c\})$ and $s \in (m_2')^{-1}(\{c\}) \setminus \mu(m_1)^{-1}(\{c\})$ such that $s \succ_c s'$. If these hold then (c, s) is a period 2 blocking pair such that

- $c \in \mathcal{C}(m_1)$ and $s \in \mathcal{S}(m_1)$,
- $m_2''(s) = c$, and
- $c \succ_s \mu(m_1)(s)$ and $(m_2'')^{-1}(\{c\}) \succ_c \mu(m_1)^{-1}(\{c\})$

where $(m_2'')^{-1}(\{c\}) = \big(\mu(m_1)^{-1}(\{c\}) \setminus \{s'\} \cup \{s\}\big)$.

Now we must show the two facts. Note that, since $m_2' \succ_c \mu(m_1)$, $(m_2')^{-1}(\{c\}) \neq \mu(m_1)^{-1}(\{c\})$. Thus either $(m_2')^{-1}(\{c\}) \subset \mu(m_1)^{-1}(\{c\})$ or $(m_2')^{-1}(\{c\}) \not\subset \mu(m_1)^{-1}(\{c\})$. If former holds, then there exists $\tilde{s} \in \mu(m_1)^{-1}(\{c\}) \setminus (m_2')^{-1}(\{c\})$ such that $\emptyset \succ_c \tilde{s}$, by responsive and transitive preferences. Therefore $(c, \tilde{s})$ is an individually irrational pair and $(c, \emptyset)$ is a period 2 blocking pair for (m_1, μ).

Now suppose $(m_2')^{-1}(\{c\}) \not\subset \mu(m_1)^{-1}(\{c\})$. Therefore, $(m_2')^{-1}(\{c\}) \setminus \mu(m_1)^{-1}(\{c\})$ is nonempty. Moreover, since (C, S) induces m_2', it follows that $(m_2')^{-1}(\{c\}) \setminus \mu(m_1)^{-1}(\{c\}) \subseteq S$, and $\mu(m_1)^{-1}(\{c\}) \setminus (m_2')^{-1}(\{c\}) \subseteq \mathcal{S}(m_1)$. To see this, first fix $s \in (m_2')^{-1}(\{c\}) \setminus \mu(m_1)^{-1}(\{c\})$. If $s \notin S$ then $m_2'(s) \in \{\mu(m_1)(s), \emptyset\}$. By definition, $m_2'(s) = c \neq \mu(m_1)(s)$ and certainly $m_2'(s) = c \neq \emptyset$. As such $s \in S$. Hence, $(m_2')^{-1}(\{c\}) \setminus \mu(m_1)^{-1}(\{c\}) \subseteq S$, which implies $(m_2')^{-1}(\{c\}) \setminus \mu(m_1)^{-1}(\{c\}) \subseteq \mathcal{S}(m_1)$ as $S \subseteq \mathcal{S}(m_1)$. For the latter, notice that if $\mu(m_1)^{-1}(\{c\}) \setminus (m_2')^{-1}(\{c\}) = \emptyset$ then clearly $\mu(m_1)^{-1}(\{c\}) \setminus (m_2')^{-1}(\{c\}) \subseteq \mathcal{S}(m_1)$. Let $\mu(m_1)^{-1}(\{c\}) \setminus (m_2')^{-1}(\{c\})$ be nonempty. Fix $s' \in \mu(m_1)^{-1}(\{c\}) \setminus (m_2')^{-1}(\{c\})$. If $s' \notin \mathcal{S}(m_1)$, then $m_1(s') \neq \emptyset$, which implies $m_2'(s') = m_1(s') = \mu(m_1)(s')$, otherwise (m_1, m_2') would not be a feasible matching. This contradicts with the assumption that $s' \in \mu(m_1)^{-1}(\{c\}) \setminus (m_2')^{-1}(\{c\})$. This proves the first fact.

To show the second fact, notice that either $|\mu(m_1)^{-1}(c)| = q_c$ or $|\mu(m_1)^{-1}(c)| < q_c$. We will consider these cases separately.

Case 1: Assume $|\mu(m_1)^{-1}(c)| = q_c$. Note $|(m_2')^{-1}(\{c\}) \setminus \mu(m_1)^{-1}(\{c\})| \leq |\mu(m_1)^{-1}(\{c\}) \setminus (m_2')^{-1}(\{c\})|$. Otherwise $|(m_2')^{-1}(\{c\})| > |\mu(m_1)^{-1}(c)| = q_c$, contradicting with the fact that m_2' is a matching. Moreover, there exists $s' \in \mu(m_1)^{-1}(\{c\}) \setminus (m_2')^{-1}(\{c\})$ and $s \in (m_2')^{-1}(\{c\}) \setminus \mu(m_1)^{-1}(\{c\})$ such that $s \succ_c s'$. To see this, assume otherwise i.e. $s' \succsim_c s$ for all $s \in (m_2')^{-1}(\{c\}) \setminus \mu(m_1)^{-1}(\{c\})$ and $s' \in \mu(m_1)^{-1}(\{c\}) \setminus (m_2')^{-1}(\{c\})$. Define $\hat{S} = \mu(m_1)^{-1}(\{c\}) \cap (m_2')^{-1}(\{c\})$. Consider $s_1' \in \mu(m_1)^{-1}(\{c\}) \setminus (m_2')^{-1}(\{c\})$, and $s_1 \in (m_2')^{-1}(\{c\}) \setminus \mu(m_1)^{-1}(\{c\})$. Notice by responsive preferences $s_1' \succ_c s_1$ implies $\hat{S} \cup \{s_1'\} \succ_c \hat{S} \cup \{s_1\}$. Now consider $s_2' \in \mu(m_1)^{-1}(\{c\}) \setminus (m_2')^{-1}(\{c\})$, and $s_2 \in (m_2')^{-1}(\{c\}) \backslash \mu(m_1)^{-1}(\{c\})$ where $s_2' \neq s_1'$ and $s_2 \neq s_1$. Again, by responsive preferences and the transitivity $s_2' \succ_c s_2$ implies $\hat{S} \cup \{s_1', s_2'\} \succ_c \hat{S} \cup \{s_1, s_2\}$. By repeated application of the fact that preferences are responsive and transitive would imply $\mu(m_1)^{-1}(\{c\}) \succ_c (m_2')^{-1}(\{c\})$, which contradicts with the fact that $m_2' \succ_c \mu(m_1)$.

Thus there exists $s' \in \mu(m_1)^{-1}(\{c\}) \setminus (m_2')^{-1}(\{c\})$ and $s \in (m_2')^{-1}(\{c\}) \setminus \mu(m_1)^{-1}(\{c\})$ such that $s \succ_c s'$. Notice responsive preferences imply that $s \succ_c s'$ if and only if $\big(\mu(m_1)^{-1}(\{c\}) \setminus \{s'\} \cup \{s\}\big) \succ_c \mu(m_1)^{-1}(\{c\})$. Therefore the pair (c, s) blocks (m_1, μ) at period 2 with m_2'' where $m_2''(s) = c$ and $m_2''(s') = \emptyset$ and for any $\ddot{s} \notin \{s, s'\}$, $m_2''(\ddot{s}) = \mu(m_1)(\ddot{s})$

Case 2: Assume $|\mu(m_1)^{-1}(c)| < q_c$. Since $s \in S$, $c \succ_s \mu(m_1)(s)$. Also $s \succ_c \emptyset$, otherwise m_2' would not be individually rational. Since preferences are responsive and $s \notin \mu(m_1)^{-1}(\{c\})$, $\big(\mu(m_1)^{-1}(\{c\}) \cup \{s\}\big) \succ_c \mu(m_1)^{-1}(\{c\})$. So the pair (c, s) blocks (m_1, μ) at period 2 with m_2'' where $m_2''(s) = c$ and $m_2''(\ddot{s}) = \mu(m_1)(\ddot{s})$ for any $\ddot{s} \neq s$. □

Fix a many-to-one dynamic matching environment $\mathcal{E}^m$ and the pair (m_1, μ) on $\mathcal{E}^m$. Define the related one-to-one dynamic matching environment $\mathcal{E}^o$ and corresponding pair $(\hat{m}_1, \hat{\mu})$ on $\mathcal{E}^o$, as in page 36.

Lemma D.2. *There exists a period 2 blocking pair for (m_1, μ) in $\mathcal{E}^m$ if and only if there exists a period 2 blocking pair for corresponding $(\hat{m}_1, \hat{\mu})$ in $\mathcal{E}^o$.*

Proof of Lemma D.2. Let the pair (c, s) block (m_1, μ) in period 2. Then $c \succ_s \mu(m_1)(s)$ and either there exists $s' \in \mu(m_1)^{-1}(\{c\}) \cap \mathcal{S}(m_1)$ such that $\big(\mu(m_1)^{-1}(\{c\}) \setminus \{s'\} \cup \{s\}\big) \succ_c \mu(m_1)^{-1}(\{c\})$ or $\big(\mu(m_1)^{-1}(\{c\}) \cup \{s\}\big) \succ_c \mu(m_1)^{-1}(\{c\})$. By responsive preferences either $s \succ_c s'$ or $s \succ_c \emptyset$. We will show these cases separately, but first note that $\hat{\mu}(\hat{m}_1)(s') = x_i^c$ for some i, and also $\mu(m_1)(s) \in \{c', \emptyset\}$ for some $c' \neq c$.

Suppose the former i.e. there exists $s' \in \mu(m_1)^{-1}(\{c\}) \cap \mathcal{S}(m_1)$ such that $\big(\mu(m_1)^{-1}(\{c\}) \setminus \{s'\} \cup \{s\}\big) \succ_c \mu(m_1)^{-1}(\{c\})$, that is $s \succ_c s'$. Also $c \succ_s \mu(m_1)(s)$ and $\mu(m_1)(s) \in \{c', \emptyset\}$ implies $c \succ_s c'$. Recall from the mapping on page 36 that $c \succ_s c'$ implies $x_i^c \succ_s x_j^{c'}$ for any i and j thus $x_i^c \succ_s \hat{\mu}(\hat{m}_1)(s)$. Also $s \succ_c s'$ implies that $s \succ_{x_i^c} s' = \hat{\mu}(\hat{m}_1)^{-1}(x_i^c)$. Therefore (x_i^c, s) is a period 2 blocking pair to corresponding $(\hat{m}_1, \hat{\mu})$.

Now suppose $\big(\mu(m_1)^{-1}(\{c\}) \cup \{s\}\big) \succ_c \mu(m_1)^{-1}(\{c\})$. Then note that $|\mu(m_1)^{-1}(\{c\})| < q_c$, otherwise $|\big(\mu(m_1)^{-1}(\{c\}) \cup \{s\}\big)| > q_c$ contradicting with quota restriction. But recall from the mapping on page 36 that if $|\mu(m_1)^{-1}(\{c\})| < q_c$ then $\hat{\mu}(\hat{m}_1)^{-1}(x_k^c) = \{\emptyset\}$ for some $k \leq q_c$. Since $s \succ_c \emptyset$, $s \succ_{x_k^c} \emptyset = \hat{\mu}(\hat{m}_1)^{-1}(x_k^c)$. But also remember that $x_k^c \succ_s \hat{\mu}(\hat{m}_1)(s)$. Thus, (x_k^c, s) is a period 2 blocking pair to corresponding $(\hat{m}_1, \hat{\mu})$.

Now, let the pair (x_i^c, s) block $(\hat{m}_1, \hat{\mu})$ in period 2. Then $x_i^c \succ_s \hat{\mu}(\hat{m}_1)(s)$ and $s \succ_{x_i^c} \hat{\mu}(\hat{m}_1)^{-1}(\{x_i^c\})$. First notice it can not be the case that $\hat{\mu}(\hat{m}_1)(s) = x_j^c$ for some j. To see this, suppose it were the case that $\hat{\mu}(\hat{m}_1)(s) = x_j^c$ for some j. Recall from page 36 that $x_i^c \succ_s x_j^c$ for all $j > i$. Thus it has to be the case that $j > i$. But then $\hat{\mu}(\hat{m}_1)^{-1}(\{x_i^c\}) \succ_c \hat{\mu}(\hat{m}_1)^{-1}(\{x_j^c\}) = s$ which implies $\hat{\mu}(\hat{m}_1)^{-1}(\{x_i^c\}) \succ_{x_i^c} s$, contradicting with the fact that $s \succ_{x_i^c} \hat{\mu}(\hat{m}_1)^{-1}(\{x_i^c\})$. Thus $\hat{\mu}(\hat{m}_1)(s) \in \{x_j^{c'}, \emptyset\}$ for some c' such that $x_i^c \succ_s x_j^{c'}$. With $c \neq c'$ in mind, $x_i^c \succ_s \hat{\mu}(\hat{m}_1)(s)$ implies $c \succ_s \mu(m_1)(s)$.

Also, notice that $s \succ_{x_i^c} \hat{\mu}(\hat{m}_1)^{-1}(\{x_i^c\})$ implies that $\hat{\mu}(\hat{m}_1)^{-1}(\{x_i^c\}) \in \{s', \emptyset\}$ for some s' such that $s \succ_{x_i^c} s'$. But then $s \succ_c s'$. Since the preferences are responsive, $\big(\mu(m_1)^{-1}(\{c\}) \setminus \{s'\} \cup \{s\}\big) \succ_c \mu(m_1)^{-1}(\{c\})$ as $s \succ_c s'$. Therefore, (c, s) is a blocking pair for $(\hat{m}_1, \hat{\mu})$.

It is quite easy to adopt the proof above to the case where either $c = \emptyset$ or $s = \emptyset$, which we consider as a pairwise block. □

Lemma D.3. *Consider a many-to-one dynamic matching environment where preferences do not exhibit cycle. Let coalition A block m_1 with m_1', given stable μ at every $\tilde{m}_1$. Then A blocks m_1 with m_1'' given μ where m_1'' is defined as follows: for any $s \in S$,*

$$m_1''(s) = \begin{cases} m_1(s) & \text{if} \quad \mu(m_1')(s) = \mu(m_1)(s) \\ m_1'(s) & \quad\quad \text{otherwise} \end{cases}$$

Proof of Lemma D.3. First need to show that the coalition A can deviate from m_1 to m_1'', by knowing that A can deviate to m_1'. Notice the following, for all $s \in \mathcal{S}_1$ such that $\mu(m_1')(s) = \mu(m_1)(s)$ we have that $s \notin A$.

Assume for contradiction that A cannot induce m_1'' from m_1. If so then at least one of the followings must be true:

i. for some $s \in A$, $m_1''(s) \notin A \cup \{\emptyset\}$

ii. for some $s \notin A$ with $m_1(s) \notin A$, $m_1''(s) = m_1(s)$

iii. for some $s \notin A$ with $m_1(s) \in A$, $m_1''(s) \notin \{m_1(s), \emptyset\}$.

Since A can induce m_1' from m_1 we know that for all $s \in S$, $\mu(m_1)(s) \neq \mu(m_1')(s)$ thus $m_1''(s) = m_1'(s) \in C \cup \{\emptyset\}$, which rules i. out. Also, for all $s \notin A$ with $m_1(s) \notin A$, we have that $m_1'(s) = m_1(s) = m_1''(s)$ which rules ii. out. Now assume that iii. holds i.e. for some $s \notin A$ with $m_1(s) \in A$, $m_1''(s) \notin \{m_1(s), \emptyset\}$. But then $m_1''(s) \neq \emptyset$ and $m_1''(s) \neq m_1(s)$, which implies $s \in A$; contradiction. Thus A can induce m_1'' from m_1.

Now we show that $\mu(m_1') = \mu(m_1'')$, given stable μ. To show this, we will use a result from static matching environment by Romero-Medina and Triossi (2013) which states: the stable matching in an environment with no simultaneous cycle in the preferences is unique. Our cycle definition coincides with their simultaneous cycle notion, which allows us to adapt their result here: By Romero-Medina and Triossi (2013), both $\mu(m_1'')$, and $\mu(m_1')$ are unique, given no cycle in the preferences.[22]

First notice that $\mathcal{C}(m_1'') \cup \mathcal{S}(m_1'') \subseteq \mathcal{C}(m_1') \cup \mathcal{S}(m_1')$. Assume for contradiction that $\mu(m_1') \neq \mu(m_1'')$. This implies $\mu(m_1')(s) \neq \mu(m_1'')(s)$ for some $s \in \mathcal{S}(m_1'')$. To see this, notice that for all s such that $m_1'(s) \neq \emptyset$, $m_1'(s) = m_1''(s)$, which implies $\mu(m_1')(s) = \mu(m_1'')(s)$ for such s. Also, for all s such that $m_1'(s) = \emptyset \neq m_1(s)$, and $\mu(m_1')(s) = \mu(m_1)(s)$, we have $m_1''(s) = m_1(s)$, which implies $\mu(m_1'')(s) = \mu(m_1)(s) = \mu(m_1')(s)$. Therefore, $\mu(m_1') \neq \mu(m_1'')$ implies $\mu(m_1')(s) \neq \mu(m_1'')(s)$ for some $s \in \mathcal{S}(m_1'')$.

The unique stable matching on $\mathcal{C}(m_1'') \cup \mathcal{S}(m_1'')$ is $\mu(m_1'')$, thus $\mu(m_1')|_{\mathcal{C}(m_1'')\cup\mathcal{S}(m_1'')}$; that is the part of $\mu(m_1')$ restricted only on $\mathcal{C}(m_1'') \cup \mathcal{S}(m_1'')$, is not stable over $\mathcal{C}(m_1'') \cup \mathcal{S}(m_1'')$. Therefore there exist a pair $(c, s) \in \mathcal{C}(m_1'') \cup \mathcal{S}(m_1'')$ which blocks $\mu(m_1')|_{\mathcal{C}(m_1'')\cup\mathcal{S}(m_1'')}$. But, notice that $(c, s) \in \mathcal{C}(m_1') \cup \mathcal{S}(m_1')$, thus (c, s) is a blocking pair also for $\mu(m_1')$ over $\mathcal{C}(m_1') \cup \mathcal{S}(m_1')$, which contradicts with the stability of $\mu(m_1')$ over $\mathcal{C}(m_1') \cup \mathcal{S}(m_1')$. Thus $\mu(m_1') = \mu(m_1'')$. This finishes the proof □

Lemma D.4. *Consider a one-to-one dynamic matching environment $\mathcal{E}^o \equiv (\mathcal{X}, \mathcal{S}_1, \mathcal{S}_2; \succsim_{\mathcal{X}}, \succsim_{\mathcal{S}})$. If preferences do not exhibit any cycle we have the following: for any dynamically stable matching (m_1, μ), for any $x \in \mathcal{X}$ and $s \in \mathcal{S}$ if $s \succ_x \mu(m_1)(x)$ then $\mu(m_1)(s) \succ_s x$.*

Proof of Lemma D.4. We will prove this lemma through contradiction. Let (m_1, μ) be a dynamically stable matching in an arbitrary dynamic one-to-one matching environment. Assume for contradiction that $\mu(m_1)(s^0) = x^0$ for some $x^0 \in X$ and there exists s^1 such that $x^0 \succ_{s^1} \mu(m_1)(s^1)$ and $s^1 \succ_{x^0} s^0 = \mu(m_1)^{-1}(x^0)$. Thus we have the following: $m_1^{-1}(x^0) = s^0 \neq \emptyset$ and $s^1 \in \mathcal{S}_2$. Otherwise, (x^0, s^1) would be a pairwise block for (m_1, μ) at some period $t = 1, 2$.

Consider alternative first period matching m_1' where $m_1'(s) = m_1(s)$ if $s \neq s^0$, and $m_1'(s^0) = \emptyset$. Dynamic stability of (m_1, μ) implies that $\mu(m_1')(s^0) = x^0$. Again application of dynamic stability of (m_1, μ) implies that $\mu(m_1')(s^1) \succ_{s^1} x^0$, otherwise (x^0, s^1) would be a period-2 pairwise block for (m_1, μ) at m_1'. Let $\mu(m_1')(s^1) = x^1$. Given that μ is stable at m_1, we must have $\mu(m_1)^{-1}(x^1) \succ_{x^1} s^1$. Let $\mu(m_1)^{-1}(x^1) = s^2$. Following this argument yields the sequence:

$$\mu(m_1')(s^1) = x^1 \succ_{s^1} x^0 = \mu(m_1)(s^0), \qquad \mu(m_1')(s^i) = x^i \succ_{s^i} x^{i-1} = \mu(m_1)(s^i) \quad \text{for all} \quad i \geq 2$$
$$\text{and} \qquad \mu(m_1)^{-1}(x^i) = s^{i+1} \succ_{x^i} s^i = \mu(m_1')(x^i) \quad \text{for all} \quad i \geq 1.$$

Now we will show that $s^j = s^1$ for some $j \geq 3$. First notice that we have finitely many students

[22]Notice that $\mu(m_1'')$, and $\mu(m_1')$ are second period matchings among the available agents $\mathcal{C}(m_1'') \cup \mathcal{S}(m_1'')$, and $\mathcal{C}(m_1') \cup \mathcal{S}(m_1')$, respectively. It is also immediate to see that if the preferences over $\mathcal{C} \cup \mathcal{S}$ do not include cycle, so don't the preferences over any restricted population $C \cup S \subset \mathcal{C} \cup \mathcal{S}$.

which implies that $s^i = s^j$ for some $i \neq j$. Wlog let $j > i \geq 1$. Then we have the followings:

$$\begin{aligned} \mu(m_1)(s^j) = &\quad x^{j-1} &= \mu(m_1')(s^{j-1}) \\ &\quad\vdots \\ \mu(m_1)(s^{i+1}) = &\quad x^{i} &= \mu(m_1')(s^{i}) \\ &\quad\vdots \\ \mu(m_1)(s^2) = &\quad x^{1} &= \mu(m_1')(s^{1}) \end{aligned}$$

Notice $s^i = s^j$ implies $x^{i-1} = x^{j-1}$ which implies $s^{i-1} = s^{j-1}$. Repeated application of this argument yields that $x^{j-(i-1)} = x^1$ and $s^{j-(i-1)} = s^1$, thus we have $s^{j-i+1} = s^1$. Now, we need to show that $j - i \geq 2$. To see this, assume for contradiction that $j = i + 1$. Then $\mu(m_1)(s^i) = x^{i-1}$ and $\mu(m_1)(s^{i+1}) = x^i$ as well as $s^i = s^{i+1}$ implies that $x^i = x^{i-1}$. Then we have $\mu(m_1')(s^i) = x^i = \mu(m_1)(s^i)$ contradicting with $\mu(m_1')(s^i) \succ_{s^i} \mu(m_1)(s^i)$. Therefore $j \geq i + 2$, which gives us the preference cycle $< s^1 x^1 s^2 \cdots x^{j-i} s^{j-i+1} >$ where $s^1 = s^{j-i+1}$. This is a contradiction for our no cycles in preferences assumption, which finishes the proof. □

Consider a dynamic matching (m_1, μ) on $\mathcal{E}^m$. It is helpful to relabel the students matched under (m_1, μ) as in the following order:

$$(\mu(m_1))^{-1}(\{c\}) = (s_{c_1}, s_{c_2}, \ldots, s_{c_{q_c}}) \quad \text{where} \quad s_{c_1} \succsim_c s_{c_2} \succsim_c \cdots \succsim_c s_{c_{q_c}}$$

Note that if $|(\mu(m_1))^{-1}(\{c\})| = k < q_c$ then $s_{c_j} \equiv \emptyset$ for all $j > k$. Analogously relabel $(\mu(m_1'))^{-1}(\{c\})$ as $(s'_{c_1}, s'_{c_2}, \ldots, s'_{c_{q_c}})$ for any $m_1' \in M_1$ given μ.

Lemma D.5. *Let $(\hat{m}_1, \hat{\mu})$ be a dynamically stable matching of the related one-to-one matching market and (m_1, μ) be the corresponding dynamic matching in the original many-to-one matching market. Let $C \cup S$ block (m_1, μ) with m_1' at $t = 1$. Then we have the following: for each $c \in C$, there exists some $s'_{c_k} \in (\mu(m_1'))^{-1}(\{c\}) \setminus (\mu(m_1))^{-1}(\{c\})$ such that $s'_{c_k} \succ_c s_{c_k}$. Moreover, if preferences do not exhibit any cycle, then $\mu(m_1)(s'_{c_k}) \succ_{s'_{c_k}} c$ for any such s'_{c_k}.*

Proof of Lemma D.5. By responsive preferences, there exists some $s'_{c_k} \in \mu(m_1')(c)$ such that $s'_{c_k} \succ_c s_{c_k}$, otherwise, $(m_1, \mu) \succsim_c (m_1', \mu)$ contradicting with $c \in C$. Now we need to show $s'_{c_k} \notin \mu(m_1)(c)$ for at least one such s'_{c_k}. To show this, pick one such s'_{c_k} i.e. $s'_{c_k} \succ_c s_{c_k}$. If $s'_{c_k} \notin (\mu(m_1))^{-1}(\{c\})$, then we are done, if $s'_{c_k} \in (\mu(m_1))^{-1}(\{c\})$, then $s'_{c_k} = s_{c_j}$ for some $j < k$. Notice $s'_{c_j} \succ_c s'_{c_k} = s_{c_j}$ which implies $s'_{c_j} \succ_c s_{c_j}$. If $s'_{c_j} \notin (\mu(m_1))^{-1}(\{c\})$, then we are done, if $s'_{c_j} \in (\mu(m_1))^{-1}(\{c\})$, then $s'_{c_j} = s_{c_i}$ for some $i < j$. Following these steps, one can go up-most to the case $s'_{c_n} = s_{c_1}$ for some $n > 1$, which implies $s'_{c_1} \succ_c s_{c_1}$, and clearly $s'_{c_1} \notin (\mu(m_1))^{-1}(\{c\})$. This proves the first part.

To see $\mu(m_1)(s'_{c_k}) \succ_{s'_{c_k}} c$ first notice that $(\hat{\mu}(\hat{m}_1))^{-1}(x^c_k) = s_{c_k}$ and $s'_{c_k} \succ_c s_{c_k}$ implies $s'_{c_k} \succ_{x^c_k} s_{c_k}$. By lemma D.4 it follows that $\hat{\mu}(\hat{m}_1)(s'_{c_k}) \succ_{s'_{c_k}} x^c_k$. Given that $s'_{c_k} \notin (\mu(m_1))^{-1}(\{c\})$ we have the following $\hat{\mu}(\hat{m}_1)(s'_{c_k}) = x^{c'}_j$ for some c' such that $c' \succ_{s'_{c_k}} c$. But then $\hat{\mu}(\hat{m}_1)(s'_{c_k}) = x^{c'}_j$ implies that $\mu(m_1)(s'_{c_k}) = c' \succ_{s'_{c_k}} c$ which completes the proof □

Lemma D.6. *Let (m_1, μ) be a dynamically group stable matching of the original many-to-one matching market where preferences are strict, transitive and do not exhibit cycles. The followings hold:*

i. If $c \succ_s \mu(m_1)(s)$ then $\tilde{s} \succ_c s$ for any $\tilde{s} \in \mathcal{S}(m_1) \cap (\mu(m_1))^{-1}(\{c\})$.

ii. Let $m_1(\tilde{s}) = c$ for some $\tilde{s} \in \mathcal{S}_1$. Then $\tilde{s} \succ_c s$ for all $s \in \mathcal{S}(m_1)$ such that $c \succ_s \mu(m_1)(s)$.

Proof of Lemma D.6. We will prove part i. and ii. separately.

Proof of Part i. Let (m_1, μ) be dynamically group stable matching where $c \succ_s \mu(m_1)(s)$. Note that either $m_1(s) = \emptyset$ or $m_1(s) = \hat{c} \neq \emptyset$. If $m_1(s) = \emptyset$, then stability of μ at m_1 implies that $\tilde{s} \succ_c s$ for any $\tilde{s} \in \mathcal{S}(m_1) \cap (\mu(m_1))^{-1}(\{c\})$.

Let $m_1(s) = \hat{c} \neq \emptyset$, and assume for contradiction that $s \succ_c \underline{s}$ for some $\underline{s} \in \mathcal{S}(m_1) \cap (\mu(m_1))^{-1}(\{c\})$. Consider m'_1 where $m'_1(s) = \emptyset$ and $m'_1(\dot{s}) = m_1(\dot{s})$ for all $\dot{s} \neq s$. Dynamic group stability of (m_1, μ) implies $\mu(m_1)(s) = \hat{c} \succsim_s \mu(m'_1)(s)$ and $\mu(m_1)(\hat{c}) \succsim_{\hat{c}} \mu(m'_1)(\hat{c})$. Consider also the following matching among $\mathcal{C}(m'_1) \cup \mathcal{S}(m'_1) = \mathcal{C}(m_1) \cup \mathcal{S}(m_1) \cup \{\hat{c}, s\}$:

$$\eta_2 \equiv (\hat{c}, s) \cup \mu(m_1)|_{\mathcal{C}(m_1) \cup \mathcal{S}(m_1)}$$

Notice η_2 is not stable over $\mathcal{C}(m'_1) \cup \mathcal{S}(m'_1)$ since (c, s) is a pairwise block; that is, $c \succ_s \eta_2(s)$ and $s \succ_c \underline{s} \in (\eta_2)^{-1}(\{c\})$ implies $\{c, s\}$ blocks η_2. The Strong Stability Property (Demange, Gale, and Sotomayor) implies the following: there exists a stable matching $\tilde{\eta}_2$ over $\mathcal{C}(m'_1) \cup \mathcal{S}(m'_1)$ such that $\tilde{\eta}_2(s) \succsim_s \eta_2(s) = \mu(m_1)(s)$ and $\hat{s} \succsim_c \underline{s} \in (\eta_2)^{-1}(\{c\})$ for all $\hat{s} \in (\tilde{\eta}_2)^{-1}(\{c\})$.

But notice that there is unique stable matching over $\mathcal{C}(m'_1) \cup \mathcal{S}(m'_1)$; therefore

$$\tilde{\eta}_2 = \mu(m'_1)|_{\{\mathcal{C}(m'_1) \cup \mathcal{S}(m'_1)\}}.$$

With $\mu(m_1)(s) = \hat{c} \succsim_s \mu(m'_1)(s)$ and $\mu(m'_1)(s) \succsim_s \mu(m_1)(s) = \hat{c}$ in mind, the strict preferences yield $\mu(m'_1)(s) = \hat{c}$.

Given $\mu(m'_1)(s) = \hat{c}$ and μ is stable at m'_1, we have that $\mu(m'_1)(\underline{s}) \neq c$ since $s \succ_c \underline{s}$ and $c \succ_s \mu(m'_1)(s)$. Therefore $\mu(m_1)|_{\{\mathcal{C}(m_1) \cup \mathcal{S}(m_1)\}} \neq \mu(m'_1)|_{\{\mathcal{C}(m_1) \cup \mathcal{S}(m_1)\}}$ and both $\mu(m_1)|_{\{\mathcal{C}(m_1) \cup \mathcal{S}(m_1)\}}$ and $\mu(m'_1)|_{\{\mathcal{C}(m_1) \cup \mathcal{S}(m_1)\}}$ are stable over $\mathcal{C}(m_1) \cup \mathcal{S}(m_1)$ contradicting with the uniqueness of stable matching under a-cyclic preferences (see Romero-Medina and Triossi (2013).). This finishes the proof of part i.

Proof of Part ii. We will use contradiction i.e. let (m_1, μ) be dynamically stable matching where $m_1(\tilde{s}) = c$ for some $\tilde{s} \in \mathcal{S}_1$ and there exists some $s \in \mathcal{S}(m_1)$ such that $c \succ_s \mu(m_1)(s)$. Assume for contradiction that $s \succ_c \tilde{s}$. We will show that there exists a preference cycle

Notice that $\hat{s} \succ_c s$ for each nonempty $\hat{s} \in \mathcal{S}(m_1) \cap (\mu(m_1))^{-1}(\{c\})$. Consider m_1' where $m_1'(\tilde{s}) = \emptyset$ and $m_1'(\dot{s}) = m_1(\dot{s})$ for all $\dot{s} \neq \tilde{s}$. Dynamic stability of (m_1, μ) implies $\mu(m_1)(\tilde{s}) \succsim_{\tilde{s}} \mu(m_1')(\tilde{s}) = c$ and $(\mu(m_1))^{-1}(\{c\}) \succsim_c (\mu(m_1'))^{-1}(\{c\})$.

Let $(\mu(m_1))^{-1}(\{c\}) \equiv m_1^{-1}(\{c\}) \cup m_{2,2}^{-1}(\{c\})$ where $m_{2,2}^{-1}(\{c\}) = \mathcal{S}(m_1) \cap (\mu(m_1))^{-1}(\{c\})$. Write $m_{2,2}^{-1}(\{c\})$ as $(s_{c_1}, s_{c_2}, \ldots, s_{c_n})$ such that $s_{c_i} \succ_c s_{c_j}$ for $i < j$ and $n < q_c$. Thus $s_{c_n} \succ_c s$ implies $|(\mu(m_1))^{-1}(\{c\})| = q_c$. Let $q_c^{m_1} = q_c - |m_1^{-1}(\{c\})| \equiv n$. Notice that $q_c^{m_1'} = q_c^{m_1} + 1$. Then define analogously $(\mu(m_1'))^{-1}(\{c\}) \equiv m_1'^{-1}(\{c\}) \cup m_{2,2}'^{-1}(\{c\})$ where $m_{2,2}'^{-1}(\{c\})$ as $(s'_{c_1}, s'_{c_2}, \ldots, s'_{c_{n+1}})$

With $m_1'^{-1}(\{c\}) \cup \{\tilde{s}\} = m_1^{-1}(\{c\})$ in mind, we have the following:

$$\begin{aligned}
(\mu(m_1))^{-1}(\{c\}) \succsim_c (\mu(m_1'))^{-1}(\{c\}) \quad &\Leftrightarrow \quad m_1^{-1}(\{c\}) \cup m_{2,2}^{-1}(\{c\}) \succsim_c m_1'^{-1}(\{c\}) \cup m_{2,2}'^{-1}(\{c\}) \\
&\Leftrightarrow \quad m_1'^{-1}(\{c\}) \cup \{\tilde{s}\} \cup m_{2,2}^{-1}(\{c\}) \succsim_c m_1'^{-1}(\{c\}) \cup m_{2,2}'^{-1}(\{c\}) \\
&\Leftrightarrow \quad \{\tilde{s}\} \cup m_{2,2}^{-1}(\{c\}) \succsim_c m_{2,2}'^{-1}(\{c\}) \\
&\Leftrightarrow \quad (s_{c_1}, s_{c_2}, \ldots, s_{c_n}, \tilde{s}) \succsim_c (s'_{c_1}, s'_{c_2}, \ldots, s'_{c_{n+1}})
\end{aligned}$$

Notice $\mu(m_1)(\tilde{s}) \succsim_{\tilde{s}} \mu(m_1')(\tilde{s})$ implies either $\mu(m_1')(\tilde{s}) = \mu(m_1)(\tilde{s}) = c$ or $c \succ_{\tilde{s}} \mu(m_1')(\tilde{s})$. If $\mu(m_1')(\tilde{s}) = \mu(m_1)(\tilde{s}) = c$ then $\mu(m_1')|_{\{\mathcal{C}(m_1) \cup \mathcal{S}(m_1)\}}$ is also stable over $\mathcal{C}(m_1) \cup \mathcal{S}(m_1)$. Since there is a unique stable matching over $\mathcal{C}(m_1) \cup \mathcal{S}(m_1)$ it must be that $\mu(m_1')|_{\{\mathcal{C}(m_1) \cup \mathcal{S}(m_1)\}} = \mu(m_1)|_{\{\mathcal{C}(m_1) \cup \mathcal{S}(m_1)\}}$. But notice $s \succ_c \tilde{s}$ where $\tilde{s} \in \mathcal{S}(m_1') \cap (\mu(m_1'))^{-1}$ and $c \succ_s \mu(m_1')(s)$. Thus (c, s) forms a pairwise block to μ at m_1', contradicting with stability of μ. Therefore $\mu(m_1')(\tilde{s}) = \mu(m_1)(\tilde{s}) = c$ is not possible.

Now assume that $c \succ_{\tilde{s}} \mu(m_1')(\tilde{s})$. Since μ is stable at m_1', we have the following: $s'_{c_{n+1}} \succ_c \tilde{s}$. With $(s_{c_1}, s_{c_2}, \ldots, s_{c_n}, \tilde{s}) \succsim_c (s'_{c_1}, s'_{c_2}, \ldots, s'_{c_{n+1}})$ in mind, $s'_{c_{n+1}} \succ_c \tilde{s}$ and responsive preferences imply that there exists $s_{c_i} \in m_{2,2}^{-1}(\{c\}) \setminus m_{2,2}'^{-1}(\{c\})$ such that $s_{c_i} \succ_c s'_{c_i}$.

Initial step: $s_{c_i} \notin m_{2,2}'^{-1}(\{c\})$, $s'_{c_i} \in m_{2,2}'^{-1}(\{c\})$ and $s_{c_i} \succ_c s'_{c_i}$ together imply $m'_{2,2}(s_{c_i}) \succ_{s_{c_i}} c = m_{2,2}(s_{c_i})$. Let $c^1 \equiv m'_{2,2}(s_{c_i})$. But $c_1 \succ_{s_{c_i}} c$ and $m_{2,2}(s_{c_i}) = c$ implies that $s_{c^1} \succ_{c^1} s_{c_i}$ for all $s_{c^1} \in m_{2,2}^{-1}(\{c^1\})$. Thus $|(\mu(m_1))^{-1}(\{c^1\})| = q_{c^1}$. Also note that $q_{c^1}^{m_1'} = q_{c^1}^{m_1}$.

Given that $|(\mu(m_1))^{-1}(\{c^1\})| = q_{c^1}$ and $q_{c^1}^{m_1'} = q_{c^1}^{m_1}$, we have the following: $|m_{2,2}'^{-1}(\{c^1\})| \leq |m_{2,2}^{-1}(\{c^1\})| = q_{c^1}^{m_1}$. Since $s_{c_i} \in m_{2,2}'^{-1}(\{c^1\}) \setminus m_{2,2}^{-1}(\{c^1\})$ there exists some $s_{c^1_j} \in m_{2,2}^{-1}(\{c^1\}) \setminus m_{2,2}'^{-1}(\{c^1\})$. But then $s_{c^1_j} \succ_{c^1} s_{c_i}$.

$s_{c^1_j} \notin m_{2,2}'^{-1}(\{c^1\})$, $s_{c_i} \in m_{2,2}'^{-1}(\{c^1\})$ and $s_{c^1_j} \succ_{c^1} s_{c_i}$ together imply that $m'_{2,2}(s_{c^1_j}) \succ_{s_{c^1_j}} c^1 = m_{2,2}(s_{c^1_j})$. Let $c^2 \equiv m'_{2,2}(s_{c^1_j})$. Note that we are back to initial step.

We can repeat this only finitely many times. Thus following the logic from lemma D.4, $c^i = c^j$ for some $i \leq j - 2$, which leads to a preference cycle. It is a contradiction to our assumption of a-cyclic preferences. Therefore $c \succ_{\tilde{s}} \mu(m_1')(\tilde{s})$ is not possible either.

With $\mu(m_1)(\tilde{s}) \succsim_{\tilde{s}} \mu(m_1')(\tilde{s})$ being not possible, we have that $\mu(m_1')(\tilde{s}) \succ_{\tilde{s}} \mu(m_1)(\tilde{s})$, contradicting with dynamic stability of (m_1, μ). This finishes the proof of lemma D.6ii. □

Lemma D.7. *Let (m_1, μ) be a dynamically group stable matching of the original many-to-one matching market where preferences are strict, transitive and do not exhibit cycles. If $c \succ_s \mu(m_1)(s)$ then $\tilde{s} \succ_c s$ for any $\tilde{s} \in (\mu(m_1))^{-1}(\{c\})$.*

Proof of Lemma D.7. Let $c \succ_s \mu(m_1)(s)$ and pick some $\tilde{s} \in (\mu(m_1))^{-1}(\{c\})$. First notice that if $\tilde{s} \in \mathcal{S}(m_1)$, then lemma D.6i. implies $\tilde{s} \succ_c s$. Let $\mu(m_1)(s) = \hat{c}$ where $c \succ_s \hat{c}$. Either $s \in \mathcal{S}(m_1)$ or $s \notin \mathcal{S}(m_1)$ i.e. $m_1(s) = \hat{c} \neq \emptyset$. If $s \in \mathcal{S}(m_1)$ then lemma D.6ii. implies $\tilde{s} \succ_c s$.

Therefore the only case to show is the following: $\dot{s} \succ_c s$ for each $\dot{s} \in \mathcal{S}_1$ such that $m_1(\dot{s}) = c$ and $m_1(s) = \hat{c} \neq \emptyset$. Assume for contradiction that $s \succ_c \tilde{s}$ for some $\tilde{s}$ where $m_1(\tilde{s}) = c$. We will show that there exists a preference cycle in this case. For that, we will make use of the steps introduced in the proof of lemma D.6ii.

Now consider m_1' where $m_1'(\tilde{s}) = \emptyset$, $m_1'(s) = c$ and $m_1'(\dot{s}) = m_1(\dot{s})$ for all $\dot{s} \notin \{s, \tilde{s}\}$. Since $\mu(m_1')(s) = c \succ_s \mu(m_1)(s) = \hat{c}$, dynamic stability of (m_1, μ) implies the following: $(\mu(m_1))^{-1}(\{c\}) \succsim_c (\mu(m_1'))^{-1}(\{c\})$. Note also the followings: $\mathcal{C}(m_1') = \mathcal{C}(m_1) \cup \{\hat{c}\}$ and $\mathcal{S}(m_1') = \mathcal{S}(m_1) \cup \{\tilde{s}\}$ with $q_{\hat{c}}^{m_1'} = q_{\hat{c}}^{m_1} + 1$ and $q_{\dot{c}}^{m_1'} = q_{\dot{c}}^{m_1}$ for all $\dot{c} \neq \hat{c}$; where $q_c^{m_1} = q_c - |m_1^{-1}(\{c\})|$.

Since $m_1'^{-1}(\{c\}) \backslash m_1^{-1}(\{c\}) = \{s\}$ and $m_1^{-1}(\{c\}) \backslash m_1'^{-1}(\{c\}) = \{\tilde{s}\}$, $s \succ_c \tilde{s}$ together with responsive preferences imply that $m_1'^{-1}(\{c\}) \succ_c m_1^{-1}(\{c\})$. But then $(\mu(m_1))^{-1}(\{c\}) \succsim_c (\mu(m_1'))^{-1}(\{c\})$ implies that $m_{2,2}^{-1}(\{c\}) \succ_c m_{2,2}'^{-1}(\{c\})$, where $m_{2,2}^{-1}(\{c\}) = \mathcal{S}(m_1) \cap (\mu(m_1))^{-1}(\{c\})$ and analogously $m_{2,2}'^{-1}(\{c\})$. Notice that dynamic group stability of (m_1, μ) implies the following: $m_{2,2}$ and $m_{2,2}'$ are stable over $\mathcal{C}(m_1) \cup \mathcal{S}(m_1)$ and $\mathcal{C}(m_1') \cup \mathcal{S}(m_1')$, respectively.

As in the proof of lemma D.6.ii. relabel $m_{2,2}^{-1}(\{c\}) = (s_{c_1}, \cdots, s_{c_n})$ and $m_{2,2}'^{-1}(\{c\}) = (s_{c_1}', \cdots, s_{c_n}')$ where $n < q_c$. Since preferences are responsive, $m_{2,2}^{-1}(\{c\}) \succ_c m_{2,2}'^{-1}(\{c\})$ implies that there exists $s_{c_i} \in m_{2,2}^{-1}(\{c\}) \setminus m_{2,2}'^{-1}(\{c\})$ such that $s_{c_i} \succ_c s_{c_i}'$. Since $s_{c_i} \succ_c s_{c_i}'$, and $s_{c_i} \notin m_{2,2}'^{-1}(\{c\})$, stability of $m_{2,2}'$ implies that $m_{2,2}'(s_{c_i}) \succ_{s_{c_i}} c$. Notice $s_{c_i} \neq \tilde{s}$ since $m_{2,2}(s_{c_i}) = c$ while $m_1(\tilde{s}) = c$

Case 1: $m_{2,2}'(s_{c_i}) = \hat{c} = m_1(s)$. Thus we have $\hat{c} \succ_{s_{c_i}} \mu(m_1)(s_{c_i})$. Then by lemma D.6.ii. $s \succ_{\hat{c}} s_{c_i}$. Also lemma D.6.i. implies that $s_{c_i} \succ_c s$. Therefore we have the following preference cycle: $< c, s_{c_i}, \hat{c}, s, c >$ contradicting with a-cyclic preferences assumption.

Case 2: $m_{2,2}'(s_{c_i}) = c^1 \neq \hat{c}$. Notice $c^1 \succ_{s_{c_i}} m_{2,2}(s_{c_i}) = c$ implies $s^{c^1} \succ_{c^1} s_{c_i}$ for all $s^{c^1} \in m_{2,2}^{-1}(\{c^1\})$. Thus $|m_{2,2}^{-1}(\{c^1\})| = q_{c^1}^{m_1}$ and $|m_{2,2}'^{-1}(\{c^1\})| \leq |m_{2,2}^{-1}(\{c^1\})|$. Given that $s_{c_i} \in m_{2,2}'^{-1}(\{c^1\}) \setminus m_{2,2}^{-1}(\{c^1\})$, there exists $s_j^{c^1} \in m_{2,2}^{-1}(\{c^1\}) \setminus m_{2,2}'^{-1}(\{c^1\})$ where $s_j^{c^1} \succ_{c^1} s_{c_i}$.

With μ being stable at m_1' in mind, $s_j^{c^1} \succ_{c^1} s_{c_i}$ implies that $m_{2,2}'(s_j^{c^1}) \succ_{s_j^{c^1}} c^1$. We have following three sub-cases:

i. $m_{2,2}'(s_j^{c^1}) = \hat{c}$. Thus we have $\hat{c} \succ_{s_j^{c^1}} c^1 = \mu(m_1)(s_j^{c^1})$. Then by lemma D.6.ii. $s \succ_{\hat{c}} s_j^{c^1}$. By our assumption: $c \succ_s \hat{c}$. Also lemma D.6.i. implies that $s_{c_i} \succ_c s$. Again by assumption in case 2: $c^1 \succ s_{c_i} c$ and $s_j^{c^1} \succ_{c^1} s_{c_i}$. This stream leads to the following preference cycle: $< s_j^{c^1}, \hat{c}, s, c, s_{c_i}, c^1, s_j^{c^1} >$, contradicting with a-cyclic preferences assumption.

ii. $m'_{2,2}(s_j^{c^1}) = c$. Thus we have $c \succ_{s_j^{c^1}} c^1 = \mu(m_1)(s_j^{c^1})$. Since μ is stable at m_1, $s_{c_i} \succ_c s_j^{c^1}$. But case 2 assumes that $c^1 \succ_{s_{c_i}} m_{2,2}(s_{c_i}) = c$ and $s_j^{c^1} \succ_{c^1} s_{c_i}$. This results with the following preference cycle: $< s_j^{c^1}, c, s_{c_i}, c^1, s_j^{c^1} >$ contradicting with a-cyclic preferences.

iii. $m'_{2,2}(s_j^{c^1}) = c^2 \notin \{\hat{c}, c\}$. Note that we are back to **Case 2**. Therefore repeating the argument yields a preference cycle for every possible case as there is only finitely many agents present in the environment.

This finishes the proof of lemma D.7 □

D.2 Proof of Proposition 4.2

Proof of Proposition 4.2. We will prove part i. through a contradiction. The proof of part ii. is via a counterexample.

Proof of part i.

Assume $q_c = 1$ for all $c \in \mathcal{C}$. Let (m_1, μ) be dynamically pairwise stable matching. Assume that it is not dynamically group stable. Then, there exists some coalition (C, S) and some $t = 1, 2$ such that (C, S) blocks (m_1, μ) with m'_t. First suppose $t = 2$. Then by lemma D.1, there exists some period 2 blocking pair for (m_1, μ), contradicting with the fact that (m_1, μ) is dynamically pairwise stable.

Now suppose $t = 1$. Thus we have;

$$(i) \quad \text{for all} \quad s \in S, \quad (m'_1, \mu) \succ_s (m_1, \mu) \quad \text{and} \quad (ii) \quad \text{for all} \quad c \in C, \quad (m'_1, \mu) \succ_c (m_1, \mu)$$

First suppose that $C = \emptyset$. Then by definition; $m'_1(s) = \emptyset$ for all $s \in S$, which implies $m'_1(s) \in \{m_1(s), \emptyset\} =$ for all $s \in \mathcal{S}_1$. But then S is a period 1 blocking coalition for (m_1, μ) with m'_1 where $m'_1(s) \in \{m_1(s), \emptyset\} =$ for all $s \in \mathcal{S}_1$, contradicting with dynamic pairwise stability of (m_1, μ). Thus $C \neq \emptyset$. Similar argument follows for the case where $S = \emptyset$. Therefore, $C \neq \emptyset \neq S$.

With $C \neq \emptyset \neq S$ in mind, dynamic pairwise stability of (m_1, μ) implies that $m'_1(s) \notin \{m_1(s), \emptyset\}$ for some $s \in \mathcal{S}_1$. Let $m'_1(s) = c \neq m_1(s)$. Then obviously $s \in S$ and $c \in C$. Therefore $\mu(m'_1)(s) \succ_s \mu(m_1)(s)$ and $(\mu(m'_1)^{-1}(\{c\})) \succ_c (\mu(m_1)^{-1}(\{c\}))$. Given $q_c = 1$ and $m'_1(s) = c$, we have the followings: $c \succ_s \mu(m_1)(s)$ and $s \succ_c (\mu(m_1)^{-1}(\{c\}))$. Thus $(c, s) \in \mathcal{C} \times \mathcal{S}_1$ is a period 1 blocking pair for (m_1, μ), contradicting with dynamic pairwise stability of (m_1, μ).

Let (m_1, μ) be dynamically group stable. Obviously (m_1, μ) is dynamically pairwise stable as well.

Proof of part ii.

The proof of part ii. is through a counterexample. Consider our illustrative example on page 7. Here is a dynamically group stable matching in that environment: 7.

$$m_1 = \{(c_1; s_4, s_5)\}, \quad \mu(\tilde{m}_1) = \mu^{\mathcal{S}}(\tilde{m}_1) \quad \forall \tilde{m}_1 \in M_1$$

where $\mu^{\mathcal{S}}$ denotes the student offering deferred acceptance algorithm, given $\tilde{m}_1$. Notice that, $\mu^{\mathcal{S}}$ yields stable matching in the second period for any given first period matching $\tilde{m}_1$. Thus, there is no period-2 blocking coalition for (m_1, μ). One needs to check whether there exists period-1 blocking coalition for (m_1, μ). At m_1, we have the following second period matching market: $\mathcal{C}(m_1) = \{c_1, c_2\}$ with $q_{c_1}^{m_1} = 1$ and $q_{c_2}^{m_1} = 2$ where $q_c^{m_1}$ denotes the left over quota for c in period 2, given m_1. Similarly, $\mathcal{S}(m_1) = \{s_1, s_2, s_3, s_4\}$. Given the preference relation by Table 1 and Table 2, there is a unique stable matching;

$$\mu(m_1) = \{(c_1; s_4, s_5, s_6), (c_2; s_1, s_2), (\emptyset, s_3)\}.$$

Notice that, for any other first period matching $\tilde{m}_1 \neq m_1$, $\mu(\tilde{m}_1)(s_6) = c_2$. And thus, responsive preferences imply $\{s_1, s_2, s_3\} \succsim_{c_1} (\mu(\tilde{m}_1))^{-1}(\{c_1\})$ for any $\tilde{m}_1 \neq m_1$. Therefore c_1 is not willing to be a part of any first period blocking coalition as $\{s_4, s_5, s_6\} \succ_{c_1} \{s_1, s_2, s_3\}$. Similarly, $\{s_1, s_2\} \succsim_{c_2} (\mu(\tilde{m}_1))^{-1}(\{c_2\})$ for any $\tilde{m}_1 \in M_1$, by responsive preferences. Thus, c_2 is not willing to be a part of any first period blocking coalition either.

Since neither college is willing to join any first period blocking coalition, s_1, s_2, and s_3 are not able to change first period matching. Although s_4 and s_5 can wait to be matched at period 2, for each $s \in \{s_4, s_5\}$, $\mu(m_1) \succsim_s \mu(\tilde{m}_1)$ for any $\tilde{m}_1 \in M_1$. Therefore, neither s_4 nor s_5 is willing to join any first period blocking coalition. Hence, (m_1, μ) is dynamically group stable.

Note that (c_1, s_i) for $i = 1, 2$ is a period 1 pairwise block for (m_1, μ). Thus (m_1, μ) is not dynamically pairwise stable.

Now we find a dynamically pairwise stable matching in this environment and show that it is not dynamically group stable. Consider $(\tilde{m}_1, \mu)$; given by,

$$\tilde{m}_1 = \emptyset, \quad \mu(\dot{m}_1) = \mu^{\mathcal{S}}(\dot{m}_1) \quad \forall \dot{m}_1 \in M_1$$

where, as before, $\mu^{\mathcal{S}}$ denotes the student offering deferred acceptance algorithm, given $\dot{m}_1$. Similar to the case above, there is no period-2 blocking pair for $(\tilde{m}_1, \mu)$. Notice $s \succ_{c_1} s'$ for any $s \in (\mu(\tilde{m}_1))^{-1}(\{c_1\})$, for each $s' \in \{s_4, s_5\}$. Similarly, $\mu(\tilde{m}_1)(s) \succ_s c_2$ for each $s \in \{s_1, s_2, s_3\}$.

Lastly, $\emptyset \succ_{c_2} s'$ for each $s' \in \{s_4, s_5\}$. Thus, there is no period 1 blocking pair for $(\tilde{m}_1, \mu)$ either. Therefore, $(\tilde{m}_1, \mu)$ is dynamically pairwise stable.

Note that, $\{c_1, s_4, s_5\}$ is a period 1 blocking coalition for $(\tilde{m}_1, \mu)$ with m_1' where $m_1' = \{(c_1; s_4, s_5)\}$. Thus $(\tilde{m}_1, \mu)$ is not dynamically group stable.

Therefore, pairwise dynamic stability is neither necessary nor sufficient for dynamic stability in dynamic many-to-one matching markets. □

E Appendix: Proofs for Section 5.2

Fix a dynamic many-to-one matching environment $\mathcal{E}^m$ and the pair (m_1, μ) on $\mathcal{E}^m$. Define the related dynamic one-to-one matching environment $\mathcal{E}^o$ and corresponding pair $(\hat{m}_1, \hat{\mu})$ on $\mathcal{E}^o$, as in appendix C.1. Our next result provides conditions under which the set of dynamically group stable matchings of the dynamic many-to-one matching market is equivalent to that of the related dynamic one-to-one market. It is useful to form this link as results from dynamic one-to-one market can be extended to dynamic many-to-one markets.

Theorem E.1. *Fix $\mathcal{E}^m$ such that the preferences are transitive, strict, responsive and there is no preference cycle. Then, (m_1, μ) is dynamically group stable if and only if $(\hat{m}_1, \hat{\mu})$ is dynamically stable.*

E.1 Proof of Theorem E.1

Proof of Theorem E.1. We will do the proof of theorem E.1 separately for either directions via following propositions:

Proposition E.1. *Fix $\mathcal{E}^m$ such that the preferences are transitive, strict, responsive and there is no preference cycle. If $(\hat{m}_1, \hat{\mu})$ is dynamically stable then (m_1, μ) is dynamically group stable.*

Proof of Proposition E.1. Suppose $(\hat{m}_1, \hat{\mu})$ is dynamically stable, and (m_1, μ) is not dynamically stable. We will show a contradiction. Since (m_1, μ) is not dynamically stable, there exists some coalition (C, S) and some $t = 1, 2$ such that (C, S) blocks (m_1, μ) with m_t'.

First suppose $t = 2$. Then by lemma D.1, there exists some period 2 blocking pair for (m_1, μ). By lemma D.2, there exists some blocking pair for $(\hat{m}_1, \hat{\mu})$, contradicting with the fact that $(\hat{m}_1, \hat{\mu})$ is dynamically stable.

Now suppose $t = 1$. Let A denote the coalition i.e. $A = C \cup S$. Thus we have;

$$(i) \quad \text{for all} \quad s \in A \quad (m_1', \mu) \succ_s (m_1, \mu) \quad \text{and} \quad (ii) \quad \text{for all} \quad c \in A \quad (m_1', \mu) \succ_c (m_1, \mu)$$

We will proceed in two cases: first case states that no *college seat* becomes worse off for any college in the coalition, as opposed to (m_1, μ). To prove the first case, we show that there exists an equivalent block to $(\hat{m}_1, \hat{\mu})$ contradicting with dynamic stability of $(\hat{m}_1, \hat{\mu})$. The second case involves a college in the coalition for which there exists a seat that becomes strictly worse off, as opposed to (m_1, μ). We show that there has to be a preference cycle for case two to arise, contradicting with our assumption of a-cyclic preferences.

Case 1: (No Downgrade): For every $c \in A, \quad s'_{c_j} \succsim_c s_{c_j}$ for all j and $s'_{c_i} \succ_c s_{c_i}$ for some i. Now consider m''_1 where

$$m''_1(s) = \begin{cases} m_1(s) & \text{if} \quad \mu(m'_1)(s) = \mu(m_1)(s) \\ m'_1(s) & \text{otherwise} \end{cases}$$

By lemma D.3, A blocks (m_1, μ) at $t = 1$ with m''_1 where $\mu(m''_1) = \mu(m'_1)$. Now we can write the corresponding matching $\hat{m}''_1$ in $\mathcal{E}^o$. Notice we have the following $(\hat{\mu}(\hat{m}''_1))^{-1}(x^c_i) \succsim_{x^c_i} (\hat{\mu}(\hat{m}_1))^{-1}(x^c_i)$ for all i, for each $c \in A$. Consider the following coalition $\hat{A}$

$$\hat{A} \equiv \{s \in A\} \cup \{x^c_i \quad | \quad c \in A, \quad s'_{c_i} \succ_c s_{c_i}\}$$

Therefore $\hat{A}$ forms a blocking coalition to $(\hat{m}_1, \hat{\mu})$ at $t = 1$ with $\hat{m}''_1$. It is left to show that $\hat{A}$ can deviate from $\hat{m}_1$ to $\hat{m}''_1$. But it follows immediately from the fact that A can implement m''_1 and if $(\hat{m}''_1)^{-1}(x^c_j) \neq (\hat{m}_1)^{-1}(x^c_j)$ where $(\hat{m}''_1)^{-1}(x^c_j) = s'_{c_j}$, then $s'_{c_j} \succ_{x^c_j} s_{c_j} = (\hat{\mu}(\hat{m}_1))^{-1}(x^c_j)$. It is a contradiction to dynamic stability of $(\hat{m}_1, \hat{\mu})$, which finishes the prof of case 1.

Case 2: (Downgrade): For some $c \in A$, there exists j such that $s_{c_j} \succ_c s'_{c_j}$. We will show that there exists a preference cycle in this case.

By lemma D.5, there exists $s'_{c_k} \in (\mu(m'_1))^{-1}(\{c\})/(\mu(m_1))^{-1}(\{c\})$ such that $s'_{c_k} \succ_c s_{c_k}$, and $\mu(m_1)(s'_{c_k}) \succ_{s'_{c_k}} c$. Consequently, $s'_{c_k} \notin A$ follows.

Knowing that $\mu(m'_1)(s'_{c_k}) = c \neq \mu(m_1)(s'_{c_k})$ and $s'_{c_k} \notin A$, we have tha following: $m'_1(s'_{c_k}) = \emptyset$. To see this, notice that $m'_1(s'_{c_k}) \in \{m_1(s'_{c_k}), \emptyset\}$ as $s'_{c_k} \notin A$. If $m_1(s'_{c_k}) \neq \emptyset$ then $m_1(s'_{c_k}) = m'_1(s'_{c_k})$ implying that $\mu(m'_1)(s'_{c_k}) = \mu(m_1)(s'_{c_k})$. This contradicts with the fact that $s'_{c_k} \in (\mu(m'_1))^{-1}(\{c\}) \setminus (\mu(m_1))^{-1}(\{c\})$. Thus $m'_1(s'_{c_k}) = \emptyset$. First notice that (m_1, μ) has no period-2 blocking coalitions.

Initial Step: Denote $\mu(m_1)(s'_{c_k}) = c^0$. Notice $c^0 \neq \emptyset$ since $c^0 \succ_{s'_{c_k}} \mu(m'_1)(s'_{c_k}) \equiv c \succ_{s'_{c_k}} \emptyset$. Last inequality follows from the fact that (m_1, μ) has no period-2 blocking coalitions. Moreover, $|(\mu(m'_1))^{-1}(\{c^0\})| = q_{c^0}$, otherwise (c^0, s'_{c_k}) would form a second period block to μ at m'_1. Moreover, stability of μ at m'_1 implies one of the followings:

1. $|(m'_1)^{-1}(\{c^0\})| = q_{c^0}$ so c^0 is not available in period 2; that is $c^0 \notin \mathcal{C}(m'_1)$, or

2. $|(m'_1)^{-1}(\{c^0\})| < q_{c^0}$ and $s \succ_{c^0} s'_{c_k}$ for each $s \in (\mu(m'_1))^{-1}(\{c^0\}) \setminus (m'_1)^{-1}(\{c^0\})$

Notice that 2 is implied both by stability of μ at m'_1 as well as the responsive preferences.

First we will show that 1 cannot arise. To see this, assume for contradiction that 1 holds. Then $(m_1')^{-1}(\{c^0\}) = (\mu(m_1'))^{-1}(\{c^0\}) \neq (\mu(m_1))^{-1}(\{c^0\})$. Thus $|(m_1')^{-1}(\{c^0\})| = q_{c^0}$ and $(m_1')^{-1}(\{c^0\}) \neq (m_1)^{-1}(\{c^0\})$ together imply that $(m_1')^{-1}(\{c^0\}) \setminus (m_1)^{-1}(\{c^0\})$ is nonempty. Thus $c^0 \in A$. Since $c^0 \in A$, $(\mu(m_1'))^{-1}(\{c^0\}) \succ_{c^0} (\mu(m_1))^{-1}(\{c^0\})$. By lemma D.5, there exists some $s'_{c_t^0} \in (\mu(m_1'))^{-1}(\{c^0\}) \setminus (\mu(m_1))^{-1}(\{c^0\})$ such that $s'_{c_t^0} \succ_{c^0} s_{c_t^0}$ and $\mu(m_1)(s'_{c_t^0}) \succ_{s'_{c_t^0}} c^0 = \mu(m_1')(s'_{c_t^0})$, which implies $s'_{c_t^0} \notin A$. Since $s'_{c_t^0} \notin A$, we have that $m_1'(s'_{c_t^0}) = \emptyset$, which contradicts with the fact that $s'_{c_t^0} \in (\mu(m_1'))^{-1}(\{c^0\}) = (m_1')^{-1}(\{c^0\})$. Thus 1 does not hold.

Now we will show if 2 holds, then there exists a preference cycle. For that, assume 2 holds. Knowing that $|(\mu(m_1'))^{-1}(\{c^0\})| = q_{c^0}$, and $(\mu(m_1'))^{-1}(\{c^0\}) \neq (\mu(m_1))^{-1}(\{c^0\})$ since

$$s'_{c_k} \in (\mu(m_1))^{-1}(\{c^0\}) \setminus (\mu(m_1'))^{-1}(\{c^0\})$$

we have $(\mu(m_1'))^{-1}(c^0) \setminus (\mu(m_1))^{-1}(c^0) \neq \emptyset$. To see this, assume for contradiction that $(\mu(m_1'))^{-1}(\{c^0\}) \subseteq (\mu(m_1))^{-1}(\{c^0\})$, which implies $q_{c^0} = |(\mu(m_1'))^{-1}(\{c^0\})| \leq |(\mu(m_1))^{-1}(\{c^0\})|$. But $s'_{c_k} \in (\mu(m_1))^{-1}(\{c^0\}) \setminus (\mu(m_1'))^{-1}(\{c^0\})$, thus $q_{c^0} < |(\mu(m_1))^{-1}(\{c^0\})|$ which contradicts with the quota restriction for c^0. Thus $\mu(m_1')(c^0) \setminus \mu(m_1)(c_0) \neq \emptyset$. Relabel $(\mu(m_1))^{-1}(\{c^0\})$ and $(\mu(m_1'))^{-1}(\{c^0\})$ as before

$$(\mu(m_1))^{-1}(\{c^0\}) \equiv (s_{c_1^0}, s_{c_2^0}, \ldots, s_{c_{q_{c^0}}^0}) \quad \text{and} \quad (\mu(m_1'))^{-1}(\{c^0\}) \equiv (s'_{c_1^0}, s'_{c_2^0}, \ldots, s'_{c_{q_{c^0}}^0})$$

where $s_{c_i^0} \succ_{c^0} s_{c_j^0}$, and $s'_{c_i^0} \succ_{c^0} s'_{c_j^0}$ for all $i < j$. Note that $s'_{c_k} \equiv s_{c_l^0}$ for some $l \leq q_{c^0}$ since $s'_{c_k} \in (\mu(m_1))^{-1}(\{c^0\}) \setminus (\mu(m_1'))^{-1}(\{c^0\})$.

We will show that there exists some $s \in (\mu(m_1'))^{-1}(\{c^0\}) \setminus (\mu(m_1))^{-1}(\{c^0\})$ such that $s \succ_{c^0} s_{c_l^0} \equiv s'_{c_k}$. To see this, assume for contradiction that $s'_{c_k} \succ_{c^0} s$ for each $s \in (\mu(m_1'))^{-1}(\{c^0\}) \setminus (\mu(m_1))^{-1}(\{c^0\})$. Fix $s \in (\mu(m_1'))^{-1}(\{c^0\}) \setminus (\mu(m_1))^{-1}(\{c^0\})$. Notice 2 implies $m_1'(s) = c^0$. Also knowing that $m_1(s) \neq c^0$ yields $c^0 \in A$.

Since $c^0 \in A$, there exists $s'_{c_n^0} \in (\mu(m_1'))^{-1}(\{c^0\}) \setminus (\mu(m_1))^{-1}(\{c^0\})$ such that $s'_{c_n^0} \succ_{c^0} s_{c_n^0}$ by Lemma D.5. If $n \leq l$, then $s'_{c_n^0} \succ_{c^0} s_{c_n^0} \succsim_{c^0} s_{c_l^0} \equiv s'_{c_k}$, which contradicts with the assumption that $s'_{c_k} \succ_{c^0} s$ for each $s \in (\mu(m_1'))^{-1}(\{c^0\}) \setminus (\mu(m_1))^{-1}(\{c^0\})$. Now assume $n > l$. Lemma D.5 implies $\mu(m_1)(s'_{c_n^0}) \succ_{s'_{c_n^0}} c^0$. Note that $s'_{c_{0n}} \notin A$ since $\mu(m_1)(s'_{c_n^0}) \succ_{s'_{c_n^0}} \mu(m_1')(s'_{c_n^0}) \equiv c^0$. Given that $s'_{c_n^0} \notin A$, we have the followings: $m_1'(s'_{c_n^0}) = \emptyset$, and $\mu(m_1')(s'_{c_n^0}) = c^0$. But, 2 implies that $s'_{c_n^0} \succ_{c^0} s_{c_l^0} \equiv s'_{c_k}$ contradicting with our assumption that $s'_{c_k} \succ_{c_0} s$ for each $s \in \mu(m_1')(c_0) \setminus \mu(m_1)(c_0)$.

Therefore, there exists some $s \in (\mu(m_1'))^{-1}(\{c^0\}) \setminus (\mu(m_1))^{-1}(\{c^0\})$ such that $s \succ_{c^0} s_{c_l^0} \equiv s'_{c_k}$, where $s \equiv s'_{c_n^0}$. Lemma D.5 implies that $\mu(m_1)(s'_{c_n^0}) \succ_{s'_{c_n^0}} c^0$. Denote $\mu(m_1)(s'_{c_n^0}) \equiv c^1$ for some $c^1 \in \mathcal{C}$. Notice that we are back to initial step where s'_{c_k} and c^0 correspond to $s'_{c_n^0}$ and c^1.

Repeating the steps yields another $s'_{c_m^1}$ and c^2, and etc. This process gives a preference ranking as in the following way: $s'_{c_k} \succ_c s_{c_k}$, $c^0 \succ_{s'_{c_k}} c$, $s'_{c_n^0} \succ_{c^0} s'_{c_k}$, $c^1 \succ_{s'_{c_n^0}} c^0$ and etc. This indicates the string $cs'_{c_k} c^0 s'_{c_n^0} c^1 s'_{c_m^1} c^2 \cdots$ in the preferences.

Notice we can repeat this process only finitely many times as there is only finitely many colleges and students. Therefore, there exists some $j > i \geq 1$ such that $c^j = c^i$. Following the logic in the proof of lemma D.4 yields us for some $k \geq 1$, $c^k = c$ which yields the preference cycle $< cs'_{c_k} c^0 s'_{c^0_n} c^1 s'_{c^1_m} \cdots c^k >$. This finishes the proof of case 2, and thus the proof of proposition E.1 which is the first direction of theorem E.1. □

Proposition E.2. *Fix $\mathcal{E}^m$ such that the preferences are transitive, strict, responsive and there is no preference cycle. If (m_1, μ) is dynamically stable then $(\hat{m}_1, \hat{\mu})$ is dynamically stable.*

Proof of Proposition E.2. Suppose (m_1, μ) is dynamically stable, and $(\hat{m}_1, \hat{\mu})$ is not dynamically stable. We will show a contradiction. Since $(\hat{m}_1, \hat{\mu})$ is not dynamically stable, there exists some coalition (X, S) and some $t = 1, 2$ such that (X, S) blocks $(\hat{m}_1, \hat{\mu})$ with m'_t.

First suppose $t = 2$. Then by lemma D.1, there exists some period 2 blocking pair for $(\hat{m}_1, \hat{\mu})$. By lemma D.2, there exists some period 2 blocking pair for (m_1, μ), and again by lemma D.1, there exists some period 2 blocking coalition for (m_1, μ). This contradicts with the fact that (m_1, μ) is dynamically stable.

Now suppose $t = 1$. Let A denote the coalition i.e. $A = X \cup S$. Thus we have;

$$(i) \quad \text{for all} \quad s \in A, \quad (\hat{m}'_1, \hat{\mu}) \succ_s (\hat{m}_1, \hat{\mu}) \quad \text{and} \quad (ii) \quad \text{for all} \quad x \in A \quad (\hat{m}'_1, \hat{\mu}) \succ_x (\hat{m}_1, \hat{\mu}) \quad (1)$$

Notice, either there are new matches under $\hat{m}'_1$ as opposed to $\hat{m}_1$ or not. Thus, there are two cases to consider: (i) $\hat{m}'_1(s) \notin \{\hat{m}_1(s), \emptyset\}$ for some $s \in \mathcal{S}_1$, or (ii) $\hat{m}'_1(s) \in \{\hat{m}_1(s), \emptyset\}$ for all $s \in \mathcal{S}_1$.

First suppose (i) holds; that is, $\hat{m}'_1(s) \notin \{\hat{m}_1(s), \emptyset\}$ for some $s \in \mathcal{S}_1$. Then $s \in S$. Let $\hat{m}'_1(s) = x^c_j$ for some $c \in \mathcal{C}$. Thus $x^c_j \succ_s \hat{\mu}(\hat{m}_1)(s)$. Notice also that $x^c_j \in X$, thus $s \succ_{x^c_j} (\hat{\mu}(\hat{m}_1))^{-1}(x^c_j)$.

Furthermore, either $\hat{\mu}(\hat{m}_1)(s) = x^c_i$ for some $j < i$ or $\hat{\mu}(\hat{m}_1)(s) = x^{\hat{c}}_i$ for some $c \succ_s \hat{c}$. If $\hat{\mu}(\hat{m}_1)(s) = x^c_i$ for some $j < i$, the mapping on page 36 implies that $(\hat{\mu}(\hat{m}_1))^{-1}(x^c_j) \succ_{x^c_j} (\hat{\mu}(\hat{m}_1))^{-1}(x^c_i)$ contradicting with $s \succ_{x^c_j} (\hat{\mu}(\hat{m}_1))^{-1}(x^c_j)$. Thus $\hat{\mu}(\hat{m}_1)(s) = x^{\hat{c}}_i$ for some $c \succ_s \hat{c}$.

Given $c \succ_s \hat{c} = \mu(m_1)(s)$, lemma D.7 implies that $\tilde{s} \succ_c s$ for all $\tilde{s} \in (\mu(m_1))^{-1}(\{c\})$. Note the following holds: $(\hat{\mu}(\hat{m}_1))^{-1}(x^c_j) \succ_c s$. Consequently $(\hat{\mu}(\hat{m}_1))^{-1}(x^c_j) \succ_{x^c_j} s$, which is a contradiction to $s \succ_{x^c_j} (\hat{\mu}(\hat{m}_1))^{-1}(x^c_j)$; and thus, a contradiction to $x^c_j \in X$. Hence (i) does not hold.

Now suppose (ii) holds; that is, $\hat{m}'_1(s) \in \{\hat{m}_1(s), \emptyset\}$ for all $s \in \mathcal{S}_1$. Then consider m'_1 in $\mathcal{E}^m$ where $m'_1(s) = \hat{m}'_1(s)$. Notice that $\mathcal{C}(m'_1) = \mathcal{C}(m_1) \cup \{c | x^c_j \in X\}$ and $\mathcal{S}(m'_1) = \mathcal{S}(m_1) \cup S$. Consider the following matching over $\mathcal{C}(m'_1) \cup \mathcal{S}(m'_1)$;

$$\eta = \{\mu(m_1)(s) | s \in \mathcal{S}(m'_1)\}$$

Notice η is stable. To see this, assume otherwise i.e. there exists a pair (c, s) in $\mathcal{C}(m'_1) \times \mathcal{S}(m'_1)$

that blocks η, that is; $c \succ_s \mu(m_1)(s)$ and $s \succ_c \underline{s}$ for some $\underline{s} \in \mathcal{S}(m_1') \cap (\mu(m_1))^{-1}(\{c\})$. However lemma D.7 implies that if $c \succ_s \mu(m_1)(s)$ then $\tilde{s} \succ_c s$ for any $\tilde{s} \in (\mu(m_1))^{-1}(\{c\})$, thus $\underline{s} \succ_c s$. This is a contradiction. Hence η is stable over $\mathcal{C}(m_1') \cup \mathcal{S}(m_1')$.

Since η is unique stable matching in an a-cyclic preferences environment, $\mu(m_1') = \eta$ as there is no period-2 blocking coalition to (m_1, μ). Thus $\mu(m_1') = \mu(m_1)$ which implies $\hat{\mu}(\hat{m}_1') = \hat{\mu}(\hat{m}_1)$. Therefore (ii) cannot hold either, and $A = \emptyset$. This finishes the proof of proposition E.2, which is the other direction of theorem E.1. □

Thus, these two propositions simply implies theorem E.1. □

E.2 Proof of Theorem 5.1

Proof of Theorem 5.1. First assume that (m_1, μ) is dynamically group stable. Then lemma D.7 implies that $\mu(m_1)$ is stable over $\mathcal{C} \cup \mathcal{S}$.

Suppose η is a group stable matching over $\mathcal{C} \cup \mathcal{S}$. We want to show there exists a dynamically stable matching (m_1, μ) in $\mathcal{E}^m$ such that $\mu(m_1) = \eta$. To see this, consider $\hat{\eta}$ in the *related static one-to-one* market that corresponds to η. Since preferences are responsive, η is stable if and only if $\hat{\eta}$ is stable (Roth & Sotomayor, 1990).

Define $(\hat{m}_1, \hat{\mu})$ where $\hat{m}_1 = \emptyset$, and $\hat{\mu}(\emptyset) = \hat{\eta}$ and $\hat{\mu}(\tilde{m}_1)$ is some random stable matching at $\tilde{m}_1$. Clearly $(\hat{m}_1, \hat{\mu})$ is dynamically stable in the *related dynamic one-to-one* market. But proposition E.1 implies that (m_1, μ) which is induced by $(\hat{m}_1, \hat{\mu})$ is dynamically stable in $\mathcal{E}^m$ where $\mu(m_1) = \eta$, which finishes the proof. □

E.3 Proof of Proposition 5.2

As we have identified in the motivating example and above, strategic manipulation is on the table if a college prefers more spread group of students. Although there is no uncertainty here, the college who is willing to do strategic manipulation behaves as if he is a risk lover. It is so since it prefers more spread -heterogeneous- groups of students to the one that is relatively less spread -homogeneous- group of students. In this section, we generalize this insight; that is, there is room for strategic manipulation unless colleges have globally average preferences. First, we will prove couple of auxiliary lemmas:

Lemma E.1. *Let $\mathcal{E}^o$ be a dynamic one-to-one matching market where agents are perfectly patient, and $\mathcal{E}^s$ be the related static one-to-one matching market. For any stable matching η on $\mathcal{E}^s$, there exists a dynamically stable matching (m_1, μ) in $\mathcal{E}^o$ such that $\mu(m_1) = \eta$.*

Proof of Lemma E.1. The proof is constructive following the logic in the second part of the proof of proposition 5.1. □

Lemma E.1 states that any statically stable matching in one-to-one matching market can be sustained as a dynamically stable matching outcome in any dynamic version of the market. It crucially depends on the fact that everyone is perfectly patient, although it is robust to slight impatience.

Lemma E.2 proves the following: no college who has globally average preferences does not match with a student, whom it would not be matched with according to the dynamically stable matching in related one-to-one market. In the example on page 7, c_1 matches with s_4 and s_5 at period 1, because c_1 prefers $\{s_4, s_5, s_6\}$ to $\{s_1, s_2, s_3\}$. In other words, since c_1 does not have globally average preferences, there is room for strategic manipulation. Thus, absent extreme preferences, the strategic manipulation via matching with worse students at period-1 is off the table. Notice, it is not the only possible blocking coalition in the original market for what related one-to-one suggests. Some college(s) might want to break some match and simply wait for the second period. However, we show that such an urge can be eliminated for dynamically stable matchings in the related market which are statically stable at the same time. Following proposition summarizes it.

Lemma E.2. *Fix $\mathcal{E}^m$ and let colleges have globally average responsive preferences. Let $(\hat{m}_1, \hat{\mu})$ be dynamically stable matching in $\mathcal{E}^o$. If (m_1, μ) is not dynamically stable then there exists period-1 blocking coalition to (m_1, μ) with m_1' where $m_1'(s) \in \{m_1(s), \emptyset\}$ for all $s \in \mathcal{S}_1$.*

Proof of Lemma E.2. Assume that $(\hat{m}_1, \hat{\mu})$ is dynamically stable but not (m_1, μ). Then there exists a t-period blocking coalition $C \cup S$ that blocks (m_1, μ) with m_t'.

First suppose $t = 2$. Then by lemma D.1, there exists some period 2 blocking pair for (m_1, μ). By lemma D.2, there exists some blocking pair for $(\hat{m}_1, \hat{\mu})$, contradicting with the fact that $(\hat{m}_1, \hat{\mu})$ is dynamically stable.

Now suppose $t = 1$. Then we have the followings:

$$(i) \quad \text{for all} \quad s \in S \quad (m_1', \mu) \succ_s (m_1, \mu) \quad \text{and} \quad (ii) \quad \text{for all} \quad c \in C \quad (m_1', \mu) \succ_c (m_1, \mu)$$

As before, let $\hat{m}_1'$ in $\mathcal{E}^o$ correspond to m_1'. Therefore $\hat{\mu}(\hat{m}_1') \succ_s \hat{\mu}(\hat{m}_1)$ for each $s \in S$. Notice one of the followings hold:

i. $m_1'(s) \notin \{m_1(s), \emptyset\}$ for some $s \in \mathcal{S}_1$.

ii. $m_1'(s) \in \{m_1(s), \emptyset\}$ for all $s \in \mathcal{S}_1$.

First, assume i. holds; that is, there exists some $s \in \mathcal{S}_1$ such that $m_1'(s) \notin \{m_1(s), \emptyset\}$. In this case we show that there exists a college c, whose preferences does not respect to **average preferences**.

Notice that $m'_1(s) = c \neq m_1(s)$ implies that $c \in C$ and $s \in S$, which implies $c \succ_s \mu(m_1)(s)$. Thus $x^c_j \succ_s \hat{\mu}(\hat{m}_1)(s)$ for any $j = 1, \ldots, q_c$. Given $(\hat{m}_1, \hat{\mu})$ is dynamically stable, $\hat{\mu}(\hat{m}_1)(x^c_j) \succ_{x^c_j} s$ for each $j = 1, \ldots, q_c$. To see this assume otherwise i.e. $s \succ_{x^c_i} \hat{\mu}(\hat{m}_1)(x^c_i)$ for some $i \leq q_c$. But $x^c_i \succ_s \hat{\mu}(\hat{m}_1)(s)$ implies that (x^c_i, s) is a period-1 blocking pair for $(\hat{m}_1, \hat{\mu})$, contradicting with dynamic stability of $(\hat{m}_1, \hat{\mu})$. Thus, $\hat{\mu}(\hat{m}_1)(x^c_j) \succ_{x^c_j} s$ for each $j = 1, \ldots, q_c$.

Note also that $c \in C$ implies $\mu(m'_1) \succ_c \mu(m_1)$. Let $(\mu(m_1)^{-1}(\{c\}) \equiv (s_1, s_2, \ldots, s_{q_c})$ where $s_i \succ_c s_j$ for any $i < j$ - analogously denote $(\mu(m'_1)^{-1}(\{c\}) \equiv (s'_1, s'_2, \ldots, s'_{q_c})$. Notice that $\hat{\mu}(\hat{m}_1)(x^c_k) \succ_{x^c_k} s \neq \emptyset$ for each $k = 1, \ldots, q_c$; equivalently, $s_i \succ_c s$ for each $i = 1, \ldots, q_c$. First, notice that $|(\mu(m_1)^{-1}(\{c\})| = q_c$, otherwise $(x^c_{q_c}, s)$ would form a blocking pair for $(\hat{m}_1, \hat{\mu})$ at period 1. Thus $|(\mu(m_1)^{-1}(\{c\})| = q_c$. Given that $|\mu(m'_1)(c)| \leq q_c$, if $|\mu(m'_1)(c)| < k = q_c$ we have $s'_j = \emptyset$ for $j = k+1, \ldots, q_c$, as before.

Let $A \equiv (\mu(m_1)^{-1}(\{c\}) \setminus (\mu(m'_1)^{-1}(\{c\})$ and $A' \equiv (\mu(m'_1)^{-1}(\{c\}) \setminus (\mu(m_1)^{-1}(\{c\})$. Since $\mu(m'_1) \succ_c \mu(m_1)$, and c has responsive preferences, $A' \succ_c A$. Wlog, let us re-re-label A such that $A \equiv (s_1, \cdots, s_m)$ where $s_i \succ_c s_j$ for $i < j$ and note that $m \leq q_c$. Analogously $A' \equiv (s'_1, \cdots, s'_n)$. Notice that $n \leq m$. If $n < m$, re-define A' as $(s'_1, \cdots, s'_m)$ such that $s'_k = \emptyset$ for each $k = n+1, \ldots, m$ so that A and A' has the same dimension.

With A and A' in hand, we know that $s \in A'$ thus $s = s'_i$ for some $i \leq n$. Notice that $s_m \succ_c s'_i$, since $\hat{\mu}(\hat{m}_1)(x^c_j) \succ_{x^c_j} s$ for each $j = 1, \ldots, q_c$. Thus $s_i \succ_c s'_i$; even more, $s_k \succ_c s'_k$ for any $k = i, i+1, \ldots, m$.

Given that c has responsive preferences and $A' \succ_c A$, there exists some $j < i$ such that $s'_j \succ_c s_j$. Notice that there could be multiple indices $k < i$ such that $s'_k \succ_c s_k$. Let j be the minimum such index i.e. $j = \min\{k : s'_k \succ_c s_k\}$. Therefore we have the following relation:

$$A \setminus \{s_j, s_{j+1}, \ldots, s_i\} \succ_c A' \setminus \{s'_j, s'_{j+1}, \ldots, s'_i\}$$

since $s_l \succ_c s'_l$ for any $l \notin \{j, j+1, \cdots, i\}$. As c has responsive preferences and $A' \succ_c A$, it must be the case that $\{s'_j, s'_{j+1}, \ldots, s'_i\} \succ_c \{s_j, s_{j+1}, \ldots, s_i\}$. By construction, $s'_j \succ_c s_j \succ_c s_i \succ_c s'_i \succ_c \emptyset$, therefore an average responsive c has the following contradicting relation

$$\{s_j, s_{j+1}, \ldots, s_i\} \succ_c \{s'_j, s'_{j+1}, \ldots, s'_i\}$$

Therefore i. is not possible. Thus $m'_1(s) \in \{m_1(s), \emptyset\}$ for all $s \in \mathcal{S}_1$, which finishes the proof. □

Proof of Proposition 5.2. We will do a direct proof. Let η be a stable matching on $\mathcal{E}^s$, and thus it is stable over $\mathcal{C} \cup \mathcal{S}$. Consider $\hat{\eta}$ in the *related static one-to-one* market that corresponds to η. Since preferences are responsive η is stable if and only if $\hat{\eta}$ is stable (Roth & Sotomayor, 1990). Lemma E.1 implies that there exists a dynamically stable matching $(\hat{m}_1, \hat{\mu})$ in the *related dynamic* *one-to-one* market such that $\hat{\mu}(\hat{m}_1) = \hat{\eta}$.

Knowing $(\hat{m}_1, \hat{\mu})$ is dynamically stable (m_1, μ) in $\mathcal{E}^o$, we will show that there exists a dynamically group stable matching $(m_1, \tilde{\mu})$ in $\mathcal{E}^m$ such that $\tilde{\mu}(m_1) \equiv \hat{\mu}(\hat{m}_1)$. First construct (m_1, μ) as in page 36. If (m_1, μ) is dynamically group stable, then we are done. Thus, suppose (m_1, μ) is not dynamically group stable.

By lemma E.2, there exists a period-1 blocking coalition to (m_1, μ) with m_1' where $m_1'(s) \in \{m_1(s), \emptyset\}$ for all $s \in \mathcal{S}_1$. Let $C \cup S$ block (m_1, μ) with m_1'. Then we have the followings:

$$(i) \quad \text{for all} \quad s \in S, \quad (m_1', \mu) \succ_s (m_1, \mu) \quad \text{and} \quad (ii) \quad \text{for all} \quad c \in C, \quad (m_1', \mu) \succ_c (m_1, \mu)$$

Now re-define the contingent matching $\tilde{\mu}$ as follows:

$$\tilde{\mu}(m_1') = \mu(m_1) \quad \text{and} \quad \tilde{\mu}(\dot{m}_1) = \mu(\dot{m}_1) \quad \forall \dot{m}_1 \neq m_1'$$

Note that $\tilde{\mu}$ is stable over $\mathcal{C}(m_1') \cup \mathcal{S}(m_1')$. In other words, there is no $t = 2$ blocking coalition to $(m_1, \tilde{\mu})$ at m_1'. To see this, assume otherwise i.e. there is a blocking coalition to $(m_1, \tilde{\mu})$ at m_1'. Then by lemma D.1, there exists a period 2 blocking pair for $(m_1, \tilde{\mu})$ at m_1'. By lemma D.2, there exists a period 2 blocking pair for $(\hat{m}_1, \hat{\tilde{\mu}})$ at $\hat{m}_1'$.

Notice that $\tilde{\mu}(m_1') = \mu(m_1)$ implies $\hat{\tilde{\mu}}(\hat{m}_1') = \hat{\mu}(\hat{m}_1)$. But $\hat{\mu}(\hat{m}_1) = \hat{\eta}$. Thus the blocking pair for $(\hat{m}_1, \hat{\tilde{\mu}})$ at $\hat{m}_1'$ blocks $\hat{\tilde{\mu}}(\hat{m}_1') = \hat{\mu}(\hat{m}_1) = \hat{\eta}$, contradicting with stability of $\hat{\eta}$. Thus $\tilde{\mu}$ is stable over $\mathcal{C}(m_1') \cup \mathcal{S}(m_1')$.

Now, for any possible $t = 1$ blocking coalition to (m_1, μ) with m_1', redefine μ at m_1' as such i.e. $\tilde{\mu}(m_1') = \mu(m_1)$ so that and the coalition is discouraged and $(m_1, \tilde{\mu})$ is dynamically stable where $\tilde{\mu}(m_1) \equiv \hat{\mu}(\hat{m}_1) = \hat{\eta} \equiv \eta$, which finishes the proof. □

E.4 The Role of Quotas

For strategic manipulation to be successful in the original market, a commitment by the colleges to not poach students from each other is also necessary. So, commitment is another important ingredient of such manipulation. Since it is directly related to the quotas of colleges, we analyze the role that quotas play in this section.

In a many-to-one matching market, each college c is constrained in the number of students to admit with, q_c. If $q_c = 1$ for all the colleges, then we are back to standard one-to-one matching market. Notice in the illustrative example on page 7, college c_1 is not matched with s_1, s_2 or s_3 in the second period, since there is no more slot for all of them. Now, consider a slightly different scenario: example A.2 As Example A.2 illustrates, when there are abundant slots in the market, strategic manipulation does not arise. The underlying reason for that is the lack of commitment due to the high number of slots. The following proposition summarizes what is observed through this example.

Proposition E.3. *Fix $\mathcal{E}^m$ such that colleges have strict and responsive preferences. Let $\mathcal{E}^s$ be the related static many-to-one matching market. If $q_c \geq |\mathcal{S}_1 \cup \mathcal{S}_2|$ for each $c \in \mathcal{C}$, then (m_1, μ) is dynamically group stable if and only if $(\tilde{m}_1, \tilde{\mu}) = (\emptyset, \tilde{\mu}(\emptyset))$ is dynamically group stable.*

Proposition E.3 has strong premise, that is the abundant number of slots by each college. However, the number of slots announced initially and filled during the market is observable. Thus the result is useful for empirical analysis although it relies on strong assumptions.

Intuitively, in all dynamically stable matchings, students should match with their top ranked colleges for whom they are acceptable. It is because colleges have excessive number of slots. Thus, we can reduce the preference listings accordingly, which leads to acyclic preferences. Then, Proposition E.3 follows simply by referring to Theorem 5.1.

E.4.1 Proof of Proposition E.3

Lemma E.3. *Fix $\mathcal{E}^m$ such that the preferences are transitive, strict and responsive. Let $q_c \geq |\mathcal{S}_1 \cup \mathcal{S}_2|$ for each $c \in \mathcal{C}$. Fix $s \in \mathcal{S}$, and let $c = \max_{\succsim_s}\{\dot{c} \in \mathcal{C} : s \succ_{\dot{c}} \emptyset\}$. The followings hold:*

i. $\mu(m_1)(s) = c$ for any dynamically stable matching (m_1, μ) on $\mathcal{E}^m$.

ii. $\hat{\mu}(\hat{m}_1)(s) = x_j^c$ for any dynamically stable matching $(\hat{m}_1, \hat{\mu})$ on $\mathcal{E}^o$.

Proof of Lemma E.3. We will prove part i. and ii. separately.

Part i. Assume for contradiction that (m_1, μ) is a dynamically stable matching on $\mathcal{E}^m$, and $\mu(m_1)(s) = \tilde{c}$ for some $s \in \mathcal{S}$ and there exists some $c \in \mathcal{C}$ such that $c \succ_s \tilde{c}$.

Notice that, $|(\mu(m_1))^{-1}(\{c\})| \leq q_c$, since $q_c \geq |\mathcal{S}_1 \cup \mathcal{S}_2|$ and $\mu(m_1)(s) \neq c$. Notice dynamic stability of (m_1, μ) implies that $m_1(s) = \tilde{c}$. But consider $\tilde{m}_1$ where $\tilde{m}_1(s) = \emptyset$ and $\tilde{m}_1 = m_1$ otherwise. Dynamic stability of (m_1, μ) implies that $\mu(m_1)(s) = \tilde{c} \succsim_s \mu(\tilde{m}_1)(s)$. And thus, $\mu(\tilde{m}_1)(s) \neq c$. But then $|(\mu(\tilde{m}_1))^{-1}(\{c\})| \leq q_c$ for the same reasoning. And thus, (c, s) is a period 2 blocking pair for (m_1, μ). By corollary 4.1, (m_1, μ) is not dynamically stable, which is a contradiction.

Part ii. Assume for contradiction that $(\hat{m}_1, \hat{\mu})$ is a dynamically stable matching on $\mathcal{E}^o$, and $\hat{\mu}(\hat{m}_1)(s) = x_j^{\tilde{c}}$ for some $s \in \mathcal{S}$ and there exists some $c \in \mathcal{C}$ such that $c \succ_s \tilde{c}$. Notice that, $(\hat{\mu}(\hat{m}_1))^{-1}(x_k^c) = \emptyset$ for some $k \leq q_c$, since $q_c \geq |\mathcal{S}_1 \cup \mathcal{S}_2|$ and $\hat{\mu}(\hat{m}_1)(s) \neq x_i^c$ for any $1 \leq i \leq q_c$. Thus, (x_k^c, s) is a period 1 blocking pair if $s \in \mathcal{S}_1$, and a period 2 blocking pair if $s \in \mathcal{S}_2$. By Proposition 4.2, $(\hat{m}_1, \hat{\mu})$ is not dynamically stable matching. □

F Appendix: Proofs for Section 5.3 and Section 6

F.1 Proof of Proposition 5.3

Proof of Proposition 5.3. Part i. directly follows from stability of $\mu(\emptyset)$. Part ii. requires proof which is to be typed.

Proof of Part ii.

Let $c^0 \in C$. Then there exists $s \in S$ such that $m_1(s) = c^0$ and $s' \succ_{c^0} s$ for all $s' \in (\mu(\emptyset))^{-1}(\{c^0\})$ by Part i. For each $c \in \mathcal{C}$, let

$$\mu(m_1)(c) = (s_1^c, s_2^c, \dots, s_{q_c}^c) \quad \text{and} \quad \mu(\emptyset)(c) = (\sigma_1^c, \sigma_2^c, \dots, \sigma_{q_c}^c)$$

denote the matching outcomes, where $s_i^c \succ_c s_j^c$ and $\sigma_i^c \succ_c \sigma_j^c$ for any $i < j$. Since $c^0 \in C$, we have the following: $(m_1, \mu) \succ_{c^0} (\emptyset, \mu)$. Therefore, responsive preferences imply that there exists $s_{i_{c^0}}^{c^0} \in (\mu(m_1))^{-1}(\{c^0\}) \setminus (\mu(\emptyset))^{-1}(\{c^0\})$ such that $s_{i_{c^0}}^{c^0} \succ_{c^0} \sigma_{i_{c^0}}^{c^0}$. Since $\mu(\emptyset)$ is a statically stable matching, we have $\mu(\emptyset)(s_{i_{c^0}}^{c^0}) \succ_{s_{i_{c^0}}^{c^0}} c^0$. And thus, $s_{i_{c^0}}^{c^0} \in \mathcal{S}(m_1)$ since $s_{i_{c^0}}^{c^0} \notin S$.

initial step: Let $\mu(\emptyset)(s_{i_{c^0}}^{c^0}) = c^1$. Given that $c^1 \succ_{s_{i_{c^0}}^{c^0}} c^0$, $\mu(m_1)(s_{i_{c^0}}^{c^0}) = c^0$ and μ is a stable matching algorithm, $s \succ_{c^1} s_{i_{c^0}}^{c^0}$ for all $s \in (\mu(m_1))^{-1}(\{c^1\}) \setminus m_1^{-1}(\{c^1\})$. And thus, the stability of μ implies $|(\mu(\emptyset))^{-1}(c^1)| \leq |(\mu(m_1))^{-1}(c^1)| = q_{c^1}$. Moreover, there exists $s_{i_{c^1}}^{c^1} \in (\mu(m_1))^{-1}(\{c^1\}) \setminus (\mu(\emptyset))^{-1}(\{c^1\})$ such that $s_{i_{c^1}}^{c^1} \succ_{c^1} \sigma_{i_{c^1}}^{c^1}$. Again, since $\mu(\emptyset)$ is a statically stable matching, we have $\mu(\emptyset)(s_{i_{c^1}}^{c^1}) \succ_{s_{i_{c^1}}^{c^1}} c^1$. And thus, $s_{i_{c^1}}^{c^1} \in \mathcal{S}(m_1)$ since $s_{i_{c^1}}^{c^1} \notin S$.

Let $\mu(\emptyset)(s_{i_{c^1}}^{c^1}) = c^2$. Notice we are back to the initial step and we can keep iterating in the same way until we reach to a same college i.e. $c^k = c^l$ for some $k \leq l - 2$, as $k = l - 1$ is not possible.

Notice that, if $c^i = c^0$ for some $i \leq l$, then we are done. Assume for contradiction $c^i \neq c^0$ for any $i \leq l$. But notice that if $c^k = c^l$, then $c^{k-1} = c^{l-1}$, and for some $n \leq l$, $c^1 = c^n$. Therefore, assume $c^k = c^1$ for some $k > 2$ without loss of generality. Then we have the following string:

$$c^1, s_{i_{c^1}}^{c^1}, c^2, s_{i_{c^2}}^{c^2}, c^3, s_{i_{c^3}}^{c^3}, \dots, c^{k-1}, s_{i_{c^{k-1}}}^{c^{k-1}}, c^k \equiv c^1 \tag{2}$$

where $s_{i_{c^j}}^{c^j} \succ_{c^j} s_{i_{c^{j-1}}}^{c^{j-1}}$ and $c^{j+1} \succ_{s_{i_{c^j}}^{c^j}} c^j$ for each $j = 1, 2, \dots, k-1$. Also $\mu(\emptyset)(s_{i_{c^j}}^{c^j}) = c^{j+1}$ and $\mu(m_1)(s_{i_{c^j}}^{c^j}) = c^j$.

Notice that $s_{i_{c^{k-1}}}^{c^{k-1}} \in (\mu(\emptyset))^{-1}(\{c^1\}) \setminus (\mu(m_1))^{-1}(\{c^1\})$ and $s_{i_{c^k}}^{c^k} \in (\mu(m_1))^{-1}(\{c^1\}) \setminus (\mu(\emptyset))^{-1}(\{c^1\})$. Since $|(\mu(\emptyset))^{-1}(c^1)| \leq |(\mu(m_1))^{-1}(c^1)| = q_{c^1}$, for every $s \in (\mu(\emptyset))^{-1}(\{c^1\}) \setminus (\mu(m_1))^{-1}(\{c^1\})$ such that $c^1 \succ_{c^1} \mu(m_1)(s)$, there exists $s' \in (\mu(m_1))^{-1}(\{c^1\}) \setminus (\mu(\emptyset))^{-1}(\{c^1\})$ such that $s' \succ_{c^1} s$; where s and s' are exchanged via a string which forms a loop as in 2 above. Since $s_{i_{c^0}}^{c^0} \in$

$(\mu(\emptyset))^{-1}(\{c^1\}) \setminus (\mu(m_1))^{-1}(\{c^1\})$ and $c^1 \succ_{s^{c^0}_{i_{c^0}}} \mu(m_1)(s^{c^0}_{i_{c^0}}) = c^0$, there exists some $s' \succ_{c^1} s^{c^0}_{i_{c^0}}$; where s' and $s^{c^0}_{i_{c^0}}$ are exchanged via a loop as in 2. Therefore, c^0 is part of that loop, which finishes the proof. □

What Proposition 5.3 states is also in line with the empirical evidence on the job market for finance Ph.D. candidates. Every year, there are two conferences that provide placement services to the finance Ph.D. job market: the FMA in October and the AFA in January. Using survey responses of 237 former first time job market participants who sought a placement between 2007 and 2015, Volkov et. al (2016) defines several empirical measures of success at various stages of the job market. Based on their sample of 237 candidates, 45.8% of them went only to the FMA, 23.3% went only to the AFA, and 25.8% went to both. They found that a vast majority of top quintile candidates skipped the FMA and secured jobs at the AFA. The FMA appears to dominate the job market for lower quintile candidates. The percent of candidates who go to the FMA and accept a job is 34%, 38%, 69%, and 49% for the 2^{nd}, 3^{rd}, 4^{th}, and 5^{th} quintiles respectively. See Figure 2 for placement data in detail.

Figure 2: Job Market Outcomes

F.2 Proof of Proposition 6.1

Proof of Proposition 6.1. First we will show that any dynamically stable matching $(m_1, \mu_{\mathcal{S}})$ has to be statically stable as well i.e. $\mu_{\mathcal{S}}(m_1)$ is stable among $\mathcal{C} \cup \mathcal{S}$. To see this, assume otherwise i.e. $\mu_{\mathcal{S}}(m_1)$ is not statically stable, or equivalently not in the core. Then there exists c and $s \in \mathcal{S}_2$ such that $m_1(c) \neq \emptyset$, $s \succ_c m_1(c)$ and $c \succ_s \mu_{\mathcal{S}}(m_1)(s)$. Now consider the first period matching where c waits for s i.e. formally consider $m_{1,c}$ where $m_{1,c}(c) = \emptyset$ and $m_{1,c}(\cdot) = m_1(\cdot)$ otherwise. Given that $(m_1, \mu_{\mathcal{S}})$ is dynamically stable, we must have $\mu_{\mathcal{S}}(m_{1,c})(c) = m_1(c)$. Thus dynamic stability of $(m_1, \mu_{\mathcal{S}})$ implies that $\mu_{\mathcal{S}}(m_{1,c})(s) \succ_s c \succ_s \mu_{\mathcal{S}}(m_1)(s)$. Also notice that $\mu_{\mathcal{S}}(m_{1,c}) \setminus \{(c, m_1(c))\}$ is stable over $\mathcal{C}(m_1) \cup \mathcal{S}(m_1)$. Let us denote $\mu_{\mathcal{S}}(m_{1,c}) \setminus \{(c, m_1(c))\}$ as $\mu_{\mathcal{S}}(m_{1,c})|_{\mathcal{C}(m_1)\cup\mathcal{S}(m_1)}$ Thus both $\mu_{\mathcal{S}}(m_{1,c})|_{\mathcal{C}(m_1)\cup\mathcal{S}(m_1)}$ and $\mu_{\mathcal{S}}(m_1)$ is stable at m_1. $\mu_{\mathcal{S}}(m_{1,c})(s) \succ_s \mu_{\mathcal{S}}(m_1)(s)$ implies that $\mu_{\mathcal{S}}(m_1)$ is not the student optimal stable matching at m_1, contradicting with the definition of $\mu_{\mathcal{S}}$. Therefore we have the following: any dynamically stable matching $(m_1, \mu_{\mathcal{S}})$ yields a statically stable matching outcome $\mu_{\mathcal{S}}(m_1)$.

Now that we know for any dynamically stable matching $(m_1, \mu_{\mathcal{S}})$, $\mu_{\mathcal{S}}(m_1)$ is statically stable, and $(\emptyset, \mu_{\mathcal{S}})$ is always dynamically stable, we will show that there is a unique dynamically stable matching outcome for any arrivals that is equivalent to $m^{\mathcal{S}}$. For contradiction, assume there exists a dynamically stable matching $(m_1, \mu_{\mathcal{S}})$ such that $\mu_{\mathcal{S}}(m_1) \neq \mu_{\mathcal{S}}(\emptyset)$. Not that there are two cases:

1. $m_1(s) = \mu_{\mathcal{S}}(\emptyset)(s)$ for all $s \in \mathcal{S}_1$ such that $m_1(s) \neq \emptyset$. Then $\mu_{\mathcal{S}}(m_1)(s) = \mu_{\mathcal{S}}(\emptyset)(s)$ for all $s \in \mathcal{S}_2$. To see this, assume otherwise i.e. $\mu_{\mathcal{S}}(m_1)(s) \neq \mu_{\mathcal{S}}(\emptyset)(s)$ for some $\hat{s} \in \mathcal{S}_2$. Then consider the restriction of $\mu_{\mathcal{S}}$ at $\emptyset$ over $\mathcal{C}(m_1) \cup \mathcal{S}(m_1)$, that is $\mu_{\mathcal{S}}(\emptyset)|_{\{\mathcal{C}(m_1)\cup\mathcal{S}(m_1)\}}$ which is clearly stable over $\mathcal{C}(m_1) \cup \mathcal{S}(m_1)$. Thus $\mu_{\mathcal{S}}(m_1)$ can not be the student optimal stable matching at m_1 as there exists another stable matching among $\mathcal{C}(m_1) \cup \mathcal{S}(m_1)$ which $\hat{s}$ strictly prefers; that is $\mu_{\mathcal{S}}(\emptyset)|_{\{\mathcal{C}(m_1)\cup\mathcal{S}(m_1)\}} \succ_{\hat{s}} \mu_{\mathcal{S}}(m_1)$. Thus $\mu_{\mathcal{S}}(m_1)(s) = \mu_{\mathcal{S}}(\emptyset)(s)$ for all $s \in \mathcal{S}_2$. Therefore we have that $\mu_{\mathcal{S}}(m_1)(s) = \mu_{\mathcal{S}}(\emptyset)(s)$ for all $s \in \mathcal{S}$.

2. For some $\tilde{s} \in \mathcal{S}_1$, $m_1(\tilde{s}) \neq \mu_{\mathcal{S}}(\emptyset)(\tilde{s})$. Thus it has to be the case that $\mu_{\mathcal{S}}(\emptyset)(\tilde{s}) \succ_{\tilde{s}} m_1(\tilde{s})$. Then consider deviation by such $\tilde{s} \in \mathcal{S}_1$ of waits; formally consider m_1' where $m_1'(s) = \emptyset$ if $\mu_{\mathcal{S}}(\emptyset)(s) \succ_s m_1(s)$ and $m_1'(s) = m_1(s)$ otherwise. Notice that $m_1'(s) = \mu_{\mathcal{S}}(\emptyset)(s)$ for all s that is not in the coalition. Then clearly $\mu_{\mathcal{S}}(m_1')(s) = \mu_{\mathcal{S}}(\emptyset)(s)$ for all $s \in \mathcal{S}(m_1')$ (by case 1). Therefore for all $s \in \mathcal{S}_1$ such that $\mu_{\mathcal{S}}(\emptyset)(s) \succ_s m_1(s)$, we have $\mu_{\mathcal{S}}(m_1')(s) \succ_s m_1(s)$, thus they form a period 1 block contradicting with $(m_1, \mu_{\mathcal{S}})$ being dynamically stable.

Case 1 and 2 implies that any dynamically stable matching $(m_1, \mu_{\mathcal{S}})$ has equivalent outcome to $(\emptyset, \mu_{\mathcal{S}})$ which is always dynamically stable. But notice $\mu_{\mathcal{S}}(\emptyset) = m^{\mathcal{S}}$ i.e. it is the student optimal stable matching of the static environment. □

F.3 Proof of Proposition 6.2

Lemma F.1. *Let $(m_1, \mu_{\mathcal{C}})$ be college-optimal DSM. If $m_1(c) \succsim_c m^{\mathcal{C}}(c)$ for all $c \in \mathcal{C} \setminus \mathcal{C}(m_1)$ then $\mu_{\mathcal{C}}(m_1)(c) \succsim_c m^{\mathcal{C}}(c)$ for all $c \in \mathcal{C}(m_1)$.*

Proof of Lemma F.1. Let $(m_1, \mu_{\mathcal{C}})$ be *college-optimal* DSM, $m_1(c) \succsim_c m^{\mathcal{C}}(c)$ for all $c \in \mathcal{C} \setminus \mathcal{C}(m_1)$, and assume for contradiction that $m^{\mathcal{C}}(c_0) \succ_{c_0} \mu_{\mathcal{C}}(m_1)(c_0)$ for some $c_0 \in \mathcal{C}(m_1)$. Let $\mu_{\mathcal{C}}(m_1)(c_0) = s_0$ and $m^{\mathcal{C}}(c_0) = s_1$, thus $s_1 \succ_{c_0} s_0$.
Let $\mu_{\mathcal{C}}(m_1)(s_1) = c_1$. Since $(m_1, \mu_{\mathcal{C}})$ is a DSM, $c_1 \succ_{s_1} c_0$. Also notice that stability of $m^{\mathcal{C}}$ implies that $m^{\mathcal{C}}(c_1) \succ_{c_1} s_1 = \mu_{\mathcal{C}}(m_1)(c_1)$. Given our assumption, $m_1(c_1) = \emptyset$.
Notice we can repeat this argument only finitely many times as there are finitely many agents in the environment. Thus for some $k > 1$, $s_k = s_0$ and we have the following relations:

$$\begin{aligned} m^{\mathcal{C}}(c_i) = s_{i+1} \quad &\succ_{c_i} \quad s_i = \mu_{\mathcal{C}}(m_1)(c_i) \quad \forall i = 0, 1, \ldots, k-1 \\ \mu_{\mathcal{C}}(m_1)(s_j) = c_j \quad &\succ_{s_j} \quad c_{j-1} = m^{\mathcal{C}}(s_j) \quad \forall j = 1, 2, \ldots, k \end{aligned}$$

Knowing that for any c such that $m^{\mathcal{C}}(c) \succ_c \mu_{\mathcal{C}}(m_1)(c)$, $m^{\mathcal{C}}(c) \in \mathcal{S}(m_1)$, consider matching η over $\mathcal{C}(m_1) \cup \mathcal{S}(m_1)$:

$$\begin{aligned} \eta(c) &= m^{\mathcal{C}}(c) \quad \forall c \quad \text{such that} \quad m^{\mathcal{C}}(c) \succ_c \mu_{\mathcal{C}}(m_1)(c) \\ \eta(c) &= \mu_{\mathcal{C}}(m_1)(c) \quad \text{otherwise} \end{aligned}$$

Notice that η is stable over $\mathcal{C}(m_1) \cup \mathcal{S}(m_1)$. To see this assume otherwise i.e. there is a pair $\{c, s\} \in \mathcal{C}(m_1) \cup \mathcal{S}(m_1)$ that blocks η. There are two possibilities:

i. $\eta(s) = \mu_{\mathcal{C}}(m_1)(s)$ and $\eta(c) = m^{\mathcal{C}}(c)$

ii. $\eta(c) = \mu_{\mathcal{C}}(m_1)(c)$ and $\eta(s) = m^{\mathcal{C}}(s)$

If i. then $s \succ_c m^{\mathcal{C}}(c) \succ_c \mu_{\mathcal{C}}(m_1)(c)$ and $c \succ_s \mu_{\mathcal{C}}(m_1)(s)$. Thus $\{c, s\}$ blocks $(m_1, \mu_{\mathcal{C}})$ at $t = 1$, contradicting with dynamic stability of $(m_1, \mu_{\mathcal{C}})$.
If ii. then $s \succ_c \mu_{\mathcal{C}}(m_1)(c) \succsim_c m^{\mathcal{C}}(c)$ and $c \succ_s m^{\mathcal{C}}(c)$. Thus $\{c, s\}$ blocks $m^{\mathcal{C}}$, contradicting with stability of $m^{\mathcal{C}}$.
Thus η is a stable matching over $\mathcal{C}(m_1) \cup \mathcal{S}(m_1)$ and $\eta(c) \succ_c \mu_{\mathcal{C}}(m_1)(c)$ for some $c \in \mathcal{C}(m_1)$. Thus $(m_1, \mu_{\mathcal{C}})$ is not the *college-optimal* DSM. This finishes the proof. □

Proof of Proposition 6.2. Let $(m_1, \mu_{\mathcal{C}})$ be *college-optimal* DSM and assume for contradiction that $m^{\mathcal{C}}(c) \succ_c \mu_{\mathcal{C}}(m_1)(c)$ for some $c \in \mathcal{C}$. Then by lemma F.1, there exists c_0 such that $m^{\mathcal{C}}(c_0) \succ_{c_0} m_1(c_0) \neq \emptyset$.

Let $m^{\mathcal{C}}(c_0) = s_1$. If $m_1(s_1) = c_1 \neq \emptyset$ then dynamic stability of $(m_1, \mu_{\mathcal{C}})$ implies that $m_1(s_1) \succ_{s_1} c_0$ and stability of $m^{\mathcal{C}}$ implies that $m^{\mathcal{C}}(c_1) \succ_{c_1} s_1 = m_1(c_1) \neq \emptyset$.

Let $m^{\mathcal{C}}(c_1) = s_2$. If $m_1(s_2) = c_2 \neq \emptyset$ then dynamic stability of $(m_1, \mu_{\mathcal{C}})$ implies that $m_1(s_2) \succ_{s_2} c_1$ and stability of $m^{\mathcal{C}}$ implies that $m^{\mathcal{C}}(c_2) \succ_{c_2} s_2 = m_1(c_2) \neq \emptyset$.

Now consider the blocking coalition $C \equiv \{c \in \mathcal{C} | \quad m^{\mathcal{C}}(c) \succ_c m_1(c)\}$ and m_1' where

$$\begin{aligned} m_1'(c) &= \emptyset \quad \forall c \quad \text{such that} \quad m^{\mathcal{C}}(c) \succ_c m_1(c) \\ m_1'(c) &= m_1(c) \quad \text{otherwise} \end{aligned}$$

We claim that $\mu_{\mathcal{C}}(m_1')(c) \succsim_c m^{\mathcal{C}}(c) \succ_c m_1(c)$ for all $c \in C$. We will prove this by contradiction i.e. assume that $m^{\mathcal{C}}(c_0) \succ_{c_0} \mu_{\mathcal{C}}(m_1')(c_0)$ for some $c_0 \in C$.

First notice that $C \neq \emptyset$ by lemma F.1. Also note that $m^{\mathcal{C}}(c) \in \mathcal{S}(m_1')$ for each $c \in C$. To see this, assume otherwise i.e. $m^{\mathcal{C}}(c) \notin \mathcal{S}(m_1')$ for some $c \in C$. Let $m^{\mathcal{C}}(c) = s$. Then $m_1'(s) = m_1(s) = \tilde{c} \neq \emptyset$. But then dynamic stability of $(m_1, \mu_{\mathcal{C}})$ implies that $m_1(s) \succ_s c$ and stability of $m^{\mathcal{C}}$ implies that $m^{\mathcal{C}}(\tilde{c}) \succ_{\tilde{c}} s = m_1(\tilde{c}) \neq \emptyset$. Thus $\tilde{c} \in C$ which implies $s \in \mathcal{S}(m_1')$ as $m_1'(\tilde{c}) = \emptyset$, which is a contradiction. Therefore $m^{\mathcal{C}}(c) \in \mathcal{S}(m_1')$ for each $c \in C$.

Let $m^{\mathcal{C}}(c_0) = s_1$. Dynamic stability of $(m_1, \mu_{\mathcal{C}})$ implies that $\mu_{\mathcal{C}}(m_1')(s_1) \succ_{s_1} c_0$. Let $\mu_{\mathcal{C}}(m_1')(s_1) = c_1$. Then stability of $m^{\mathcal{C}}$ implies that $m^{\mathcal{C}}(c_1) \succ_{c_1} s_1 = \mu_{\mathcal{C}}(m_1')(c_1)$. But we can repeat this argument only finitely many times as there is only finitely many agents in the environment.

Following the logic in the proof of lemma F.1, define matching η over $\mathcal{C}(m_1') \cup \mathcal{S}(m_1')$:

$$\begin{aligned} \eta(c) &= m^{\mathcal{C}}(c) \quad \forall c \quad \text{such that} \quad m^{\mathcal{C}}(c) \succ_c \mu_{\mathcal{C}}(m_1')(c) \\ \eta(c) &= \mu_{\mathcal{C}}(m_1)(c) \quad \text{otherwise} \end{aligned}$$

The stability of η over $\mathcal{C}(m_1') \cup \mathcal{S}(m_1')$ with a similar argument in the proof of lemma F.1, contradicting with $(m_1, \mu_{\mathcal{C}})$ being *college-optimal* dynamically stable matching. This finishes the proof. □