Uncertain Risk and Return in Bond Markets, I

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Abstract

I use the term structure model in Cochrane and Piazzesi (2008) and construct currency market prices. The implied currency market prices are counterfactually volatile and predictable, at least with respect to commonly used predictor variables. Getting the model closer to currency market data means reducing bond risk compensation but doing so nearly eliminates predictability in bond markets. One way to generate sensible time-variation in predictable bond and currency excess returns allows the volatility of returns to be time-varying. Additional avenues that can jointly account for the stochastic properties of bond and currency excess returns are also discussed.

JEL Classification: $E43$, $E47$, $F31$, $F37$, $G12$, $G15$

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1 Introduction

Government bond markets are large and liquid, with hundreds of billions of dollars traded on an average day. [1] Fortunes rise and fall even with small changes in interest

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[1] for the United States, this is about 5 percent of annual GDP in 2010 on a daily basis!

rates, so understanding what precisely moves this market is paramount to financial market participants, central bankers, as well as academics.

For this paper, I critically examine the recent findings by Cochrane and Piazzesi (2008) in its ability to characterize the dynamic properties of returns in bond markets. I first run similar regressions as those performed in Cochrane and Piazzesi (2005) to see if there is predictability in international bond markets. I find that among industrialized countries, bond markets are as predictable as they are in the U.S. I then determine whether the mentioned authors' term structure model can deliver an empirically consistent view of exchange rates and interest rates across countries. I find that it cannot. The model is unable to do so because the discount factor used to capture time-variation in bond returns lead to incredible currency market prices. I pinpoint the underlying reasons why the model is unable to describe key features of currency market data and highlight how the amount of captured predictability in returns can directly be linked to variation in market risk prices. Long story short, there is strong circumstantial evidence that the volatility of returns is time-varying because admitting that there is not much predictability at all is inconsistent with empirical evidence in international bond markets. Unfortunately, the model specification necessarily does not allow for time-varying volatility.

Although the linear factor model accurately describes in-sample bond price dynamics, the assumption of constant conditional variances has unsettling implications for the risk-return trade-off in both bond and currency markets. Specifically, the highlighted model may not be an adequate framework to simultaneously capture predictability in two different financial markets. That is, if variation in expected excess returns in bond markets is captured by one variable, I show that there is either a plausible amount of predictability in bond prices or currencies, but not both asset prices at once.

Conditionally heteroskedastic returns can generate more sensible movements in both the pricing kernel and conditional Sharpe ratios, but it is also worth considering the possibility that the amount of measured predictability cannot be attributed to the risk factors in the stochastic discount factor alone. Included in the paper is a discussion on other avenues that can contribute to the amount of measured return predictability found in the data.

Amidst the diverse set of approaches, I choose the four-factor affine term structure model of Cochrane and Piazzesi (2008) as the foundation of my analysis because it incorporates key information in forward rates that have been shown to forecast well expected returns in bond markets. Among the industrialized countries under study, the term structure of interest rates displays similar co-movement. For this reason, it is tempting to see if returns in bond markets are predictable in other bond markets. International comparisons of may uncover useful results and perhaps reveal global forecasting factors.

The results of my paper are not necessarily negative. The limitations I highlight in the paper should not discourage us from pursuing a deeper understanding of bond and currency prices. Rather, by acknowledging that the complex nature of bond price movements cannot satisfactorily be approximated with a class of linear models, I hope to direct the literature to search for deeper explanations for predictable returns in bond and currency markets and to help reveal the underlying, macroeconomic reasons for why expected returns vary over time.

2 Literature Review

My work builds upon Fama and Bliss (1987) and Cochrane and Piazzesi (2005) who document predictable variation in expected returns using forward interest rates. I extend their analysis to other bond markets using only a subset of available forward rates, similar to Tang and Xia (2007) who also document predictable bond risk-premia in international markets under a different model specification than my own.

Similar work documenting predictable variation in bond markets within the context of a term structure model have been done by Duffee (2002, 2010) and Dai and Singleton (2002), who model returns as conditionally heteroskedastic. Adopting the more flexible specification for the market price of risk enables them to characterize time-varying risk and in doing so, account for the failure of the expectations hypothesis, making it a leading term structure model of interest rates alongside the model of Cochrane and Piazzesi (2008) my paper carefully examines. In adopting the model of Cochrane and Piazzesi (2008), I make two choices. One, I specify that returns are conditionally homoskedastic, which is in contrast to Dai and Singleton (2002). Two, I adopt a flexible specification for the market price of risk as does Duffee (2002, 2010), although I use an additional variable to do so. Both specification choices allow the model to precisely characterize risk adjusted dynamics, and my paper examines the implications of this specification choice.

Empirical research documenting predictability in currency markets dates back further than that of bond markets. The analysis in my paper tests if we can simultaneously capture predictability in bond and currency markets, especially in light of new findings by Lustig et al. (2009, 2010) who show that there is a large predictable component in currency markets for a portfolio of currencies. My analysis is similar to Backus, Foresi, and Telmer (2001) who also link bond markets to currency markets using stochastic discount factors implied by an affine model, and determine whether it can account for the forward premium anomaly.

Because the literature is rather deep, I will not be able to satisfactorily discuss the numerous important findings. I take the convenient stance and refer the reader to the references in this paper for further references. Given that my objective for this paper is to critically examine our understanding of bond markets with term structure models, I believe that the brief list of references in this paper is sufficient.

Section 3 covers the empirical findings on predictability in international bond markets. In Section 4, I explicitly link bond markets to currency markets through equilibrium asset pricing conditions and then outline the tension of fitting yields. In section 5, I discuss the affine term structure model along with a description of the implications of the model that are important for the paper. In section 6, the results explore the currency price implications of the model. In section 7, I discuss my findings in relation to the broader literature. Section 8, I conclude. Section 9 is the appendix.

3 International Predictability Regressions

3.1 Definitions

Let $P_t^{(n)}$ be the price of an n-maturity bond and $p_t^{(n)}$ be the log price. From now on, all lowercase variables are log variables. Zero-coupon bond prices are easily mapped into yields. The interest rate of an n-maturity bond is then $y_t^{(n)} = -\frac{1}{n} p_t^{(n)}$. The log return to holding an n-maturity bond for one period is $hpr_{t+1}^{(n)} = p_{t+1}^{(n-1)} - p_t^{(n)}$. The return in excess of the short rate is the realized excess return of an n-maturity bond, $rx_{t+1}^{(n)} = hpr_{t+1}^{(n)} - y_t^{(1)}$.

3.2 Data Description

The monthly frequency panel data of zero-coupon bond yield data are gathered from international bond markets, where the longest time-series available ranges from 1/1973-5/2009. Only Germany and the U.S. cover this time period. The other countries start dates are within this range and have more than 250 observations. Data availability limited the international set of data to ten countries (The U.S., Germany, Japan, Great Britain, Canada, Australia, New Zealand, Norway, and Switzerland, and Sweden). I consider exclusively the returns from an annual holding period. The longest duration studied is ten years. My analysis spans the cross-section of maturities ranging from one year up to ten years, partitioned equally twelve months apart to match the annual holding period under study. The forward rates are constructed from bond yield data using the definition given above. Only a subset of forward rates are used in the forecasting regressions, namely the one year forward rate from today to one-year ahead, from year 4 to year 5, and from year 8 to year 9. The subset of forwards is then describe by the set $\zeta := \{0 \to 1, 4 \to 5, 8 \to 9\}$. A more detailed description is relegated to the appendix.

3.3 Estimation Procedure and Findings

To construct the return forecasting factor, I stay as close as possible to Cochrane and Piazzesi (2005) with the exception that I use only a subset of forward rates constructed with country-specific data. The primary reason for using a subset of forward interest rates is practical in nature. Using the full set of available forward rates generates telltale signs of multicollinearity, and thus exposing the results to the vulnerability of how zero-coupon forward rates were interpolated from the data. The sensitivity of the results to the set of forward rates is a concern, but I mitigate many of the related issues by restricting the set of forwards to ensure that the tent-shape is preserved, a similar to the approach Tang and Xia (2007). There is no theoretical reason for ensuring that the tent-shape holds, but in practice and as discussed by Singleton (2006), it reduces the possibility that measurement error or the interpolation technique will distort results.

To be clear on notation, let $rx^{(n)}_{t+1,(j)}$ be the realized term-premia for holding onto an n year-to-maturity bond for country j one year from now. Let $\zeta = (\{0 \to 1, 4 \to 5, 8 \to 9\})$

be the subset of forwards. The set of forward rates used in the construction of the return-forecasting factor are given by $\mathbf{f}_{t,(j)} = \left(f_{t,(j)}^{(0\to1)}, f_{t,(j)}^{(4\to5)}, f_{t,(j)}^{(8\to9)}\right)$ and the coefficients $\gamma_j = \left(\left\{\gamma_j^{(0\to1)}, \gamma_j^{(4\to5)}, \gamma_j^{(8\to9)}\right\}\right)$ where the index $t,(j)$ is meant to describe the date t forward rate for country j. To actually construct the return-forecasting factor, I regress the average term-premia for a cross-section of bonds of maturity n, on the subset of country-specific forward rates:

$$\frac{1}{N}\sum_{n=1}^{N} rx_{t+1,(j)}^{(n)} = \overline{rx_{t+1,(j)}} = a_{(j)} + \gamma_{(j)}' {\cdot} \mathbf{f}_{t,(j)} + \varepsilon_{t+1} \tag{1}$$

The fitted values are then used as the "local" return-forecasting factor, i.e. the relevant information contained in forward rates given by the expression below

$$\hat{a}_{(j)} + \hat{\gamma}_{(j)}' {\cdot} \mathbf{f}_{t,(j)} = \hat{a}_{(j)} + \Sigma_\zeta \hat{\gamma}_{t,(j)} f_{t,(j)} := x_{t,(j)}. \tag{2}$$

where Σ_ζ denotes that I fit the forwards only for the relevant subset. I should emphasize that I extract the return-forecasting factor using the exact same procedure for each country. In the next step, the following time-series regressions are run to determine if there is bond predictability from holding an n-year maturity bond for one year for each country j

$$rx_{t+1,(j)}^{(n)} = \alpha_j(n) + \beta_j(n) \cdot \left(x_{t,(j)}\right) + \varepsilon_{t+1,(j)}^{(n)} \tag{3}$$

Before making international comparison, it is worth mentioning the differences in the predictable component that I find using a subset of forwards relative to the benchmark case. I find less predictability using the subset of forward rates, a considerable amount in relation to that found in Cochrane and Piazzesi (2005). To compare the amount of predictability I find for the U.S. using only a subset of forward rates to Cochrane and Piazzesi (2005, 2008) using the Fama and Bliss forward rates, I plot the two variables along with the realized bond risk-premia for an equally weighted portfolio of returns.

The Fama and Bliss forward rates seem to capture more variation in the magnitude changes in excess returns than the subset of forwards that I use, although for the most part my forecasting factor seems to capture a significant amount of variation in the *direction* of movement in bond excess returns. The difference in the amount of predictability

Figure 1: Comparing the Size of the Predictable Component

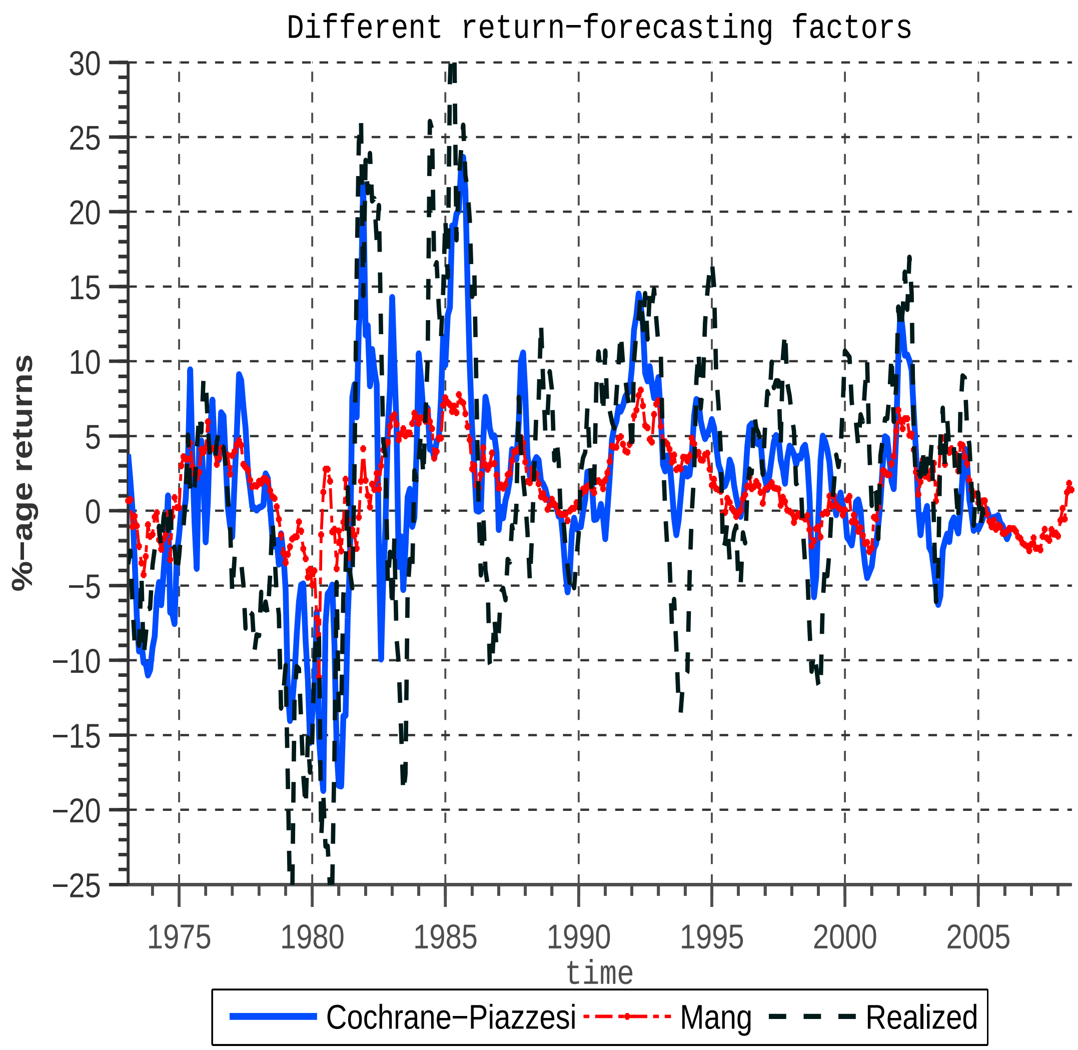

Table 1: Comparing the Predictable Component

n		2	3	4	5	6	7	8	9	10
U.S. (Cochrane-Piazzesi)	$\beta(n)$	0.17	0.33	0.47	0.60	0.73	0.85	0.97	1.08	1.19
	t-$stat$	(8.07)	(9.29)	(9.80)	(9.93)	(9.94)	(9.90)	(9.86)	(9.78)	(9.67)
	R^2	0.31	0.35	0.39	0.41	0.43	0.43	0.43	0.44	0.44
U.S. (Mang)		0.24	0.45	0.65	0.84	1.01	1.20	1.37	1.54	1.70
		(2.69)	(2.58)	(2.67)	(2.17)	(2.77)	(2.82)	(2.88)	(2.93)	(2.97)
		0.12	0.13	0.14	0.15	0.16	0.17	0.17	0.18	0.18

Note: The estimates are obtained by linear GMM with 12 lags to account for overlapping monthly data. The standard errors are Newey-West corrected. Each column n represents the regression of the 12-month ahead bond excess return from an n year-to-maturity bond on the specified subset of forward interest rates. For each group of estimates obtained using a different set of forward rates as regressors, there are three rows. The first row represents the estimated coefficient of the respective regressions. The second row displays the t-statistics in parenthesis. The third row displays the R^2 of the regression. The constant term in the regression equation is not reported as they are not significant and estimates are usually close to zero.

captured using only a subset of forwards is a reasonable concern, however the reduction in predictability will not materially alter the main results of my paper. The primary reason this is true is because it will not materially affect the underlying procedure by which market risk prices are calibrated to match the amount of predictability found in the data. These mentioned parameters are chosen to match the amount of variation of returns along the cross-section of bonds, i.e. along the time-to-maturity n dimension. Fortunately, these parameters are not that sensitive to which subset of forward rates I use to extract the return-forecasting factor, at least not when I use the subset ζ. That is, when I compare my estimates using only a subset of forwards to that of Cochrane and Piazzesi (2008) for the U.S., I find differences in the key market price of risk parameter but the magnitudes are comparable, usually in line with the degree of estimation uncertainty acknowledged by the authors. Even after accounting for the reduction of the magnitudes in the market risk prices, the central results of my paper will hold despite the use of only a subset of forwards.

I use standard (unrestricted) regression techniques to estimate the statistical model outlined by equation (3) while correcting for conditional heteroskedastic and autocorrelated errors using GMM, with 12 lags to account for overlapping monthly data for annual holding period returns. I display the results in Table 2.

A quick glance reveals that, despite some differences in when bond price data are available, the estimates display strikingly similar patterns. With few exceptions, namely

Table 2: Comparing the Predictable Component: International.

n	2	3	4	5	6	7	8	9	10
country j	$\beta(n)$ t-$stat$ R^2								
U.S. 12/71−5./09	0.24 (2.69) 0.12	0.45 (2.66) 0.13	0.65 (2.67) 0.14	0.84 (2.71) 0.15	1.01 (2.77) 0.16	1.20 (2.82) 0.17	1.37 (2.88) 0.17	1.54 (2.93) 0.18	1.70 (2.97) 0.18
Germany 1/73−5/09	0.24 (2.38) 0.09	0.46 (2.50) 0.10	0.67 (2.58) 0.11	0.85 (2.67) 0.12	1.03 (2.77) 0.12	1.20 (2.88) 0.13	1.36 (3.00) 0.13	1.52 (3.12) 0.13	1.66 (3.23) 0.13
England 1/79−5/09	0.27 (2.36) 0.08	0.52 (2.43) 0.09	0.76 (2.50) 0.10	0.98 (2.58) 0.11	1.14 (2.66) 0.11	1.26 (2.71) 0.10	1.32 (2.67) 0.09	1.36 (2.56) 0.08	1.39 (2.39) 0.07
Switzerland 1/88−5/09	0.29 (4.06) 0.32	0.51 (4.78) 0.34	0.71 (5.59) 0.36	0.88 (6.35) 0.37	1.04 (6.83) 0.38	1.19 (6.94) 0.38	1.33 (6.80) 0.37	1.45 (6.50) 0.36	1.57 (6.15) 0..35
Japan 1/85−5/09	0.19 (3.45) 0.17	0.43 (4.35) 0.20	0.67 (5.08) 0.22	0.88 (5.52) 0.23	1.05 (5.70) 0.24	1.22 (5.71) 0.24	1.37 (5.64) 0.25	1.52 (4.78) 0.25	1.65 (4.72) 0.25
Canada 1/86−5/09	0.20 (1.45) 0.05	0.38 (1.54) 0.05	0.58 (1.71) 0.07	0.79 (1.89) 0.09	1.00 (2.04) 0.10	1.20 (2.14) 0.11	1.40 (2.23) 0.12	1.61 (2.32) 0.12	1.83 (2.40) 0.14
Australia 2/87−5/09	0.17 (1.06) 0.05	0.38 (1.37) 0.08	0.60 (1.64) 0.10	0.81 (1.88) 0.12	1.01 (2.07) 0.13	1.21 (2.24) 0.14	1.41 (2.36) 0.15	1.61 (2.46) 0.16	1.80 (2.53) 0.16
New Zealand 12/92−5/09	0.24 (5.38) 0.30	0.46 (6.60) 0.35	0.67 (7.40) 0.38	0.85 (7.73) 0.39	1.02 (7.70) 0.40	1.19 (7.44) 0.40	1.36 (7.12) 0.40	1.52 (6.80) 0.39	1.68 (6.51) 0.39

See the footnotes for Table 1 for further details.

Canada and Australia, the estimated coefficients are statistically significant. Not only that, the large R^2 suggests that a relatively large amount of variability in excess returns can be statistically explained by country-specific forward rates. Statistical evidence of the predictability of returns imply that variation in the predictor variable precede movements in returns. As the results indicate, for the sample period under study, there is significant statistical evidence that variation in the forecasting factor x_t forecast movements in excess returns one year from now. Given that the variables are in logs, the regression coefficient reveals that a one percent movement in x_t corresponds to expected movements in bond excess returns to holding onto an n-year treasury by $\beta\left(n\right)$ percentage points, or

$\beta(n) \cdot 100$ basis points. For example, a one percent movement in the return-forecasting factor will on average precede a 24 basis point movement in the risk–premium for holding a 2-year treasury security in the U.S. over the time period. The empirical results also suggest that larger movements are expected from longer maturity treasuries (e.g. 170 basis point movement in the risk-premium of holding onto 10-year treasury for one year in the U.S. in response to a 1 percent movement in x_t over the sample period). In addition, the size of the regression coefficient tell us that *expected* excess returns for longer duration bonds are more sensitive to changes in the return-forecasting factor. In addition, notice the cross-country pattern of monotone increases in the regression coefficient across maturity, suggesting that the return-forecasting factor affects bonds of all maturities in a similar fashion. This lends some support to a key identifying restriction made in Cochrane and Piazzesi (2008) that random walk shocks exclusively move risk-premia, one that is used to precisely pin down the dynamics of the model. Overall, predictability in bond markets appears to be a robust, global phenomenon.

4 Linking Bond and Currency Markets

Before discussing the term structure model, I will make transparent the relationship between bond and currency markets. If markets are complete and there are no riskless arbitrage opportunities, then there exists a unique one-to-one transformation of a domestic stochastic discount factor M_{t+1} to its foreign counterpart M^*_{t+1}. In other words, there exists a unique conversion unit S_t, denominated in foreign currency per dollar, that allows the following equilibrium asset pricing condition to hold simultaneously across two different markets, as shown in the following relation:

$$\mathbb{E}_t\left[M_{t+1}R_{t+1}\right] = \mathbb{E}_t\left[M^*_{t+1}R_{t+1}\left(\frac{S_{t+1}}{S_t}\right)\right] \tag{4}$$

for some foreign stochastic return process such that that $R_{t+1}\left(\frac{S_{t+1}}{S_t}\right) = R^*_{t+1}$ for any state of the world and any time period. In plain English, we can then express returns in a foreign country as a function of dollar returns using some conversion or exchange rate. In doing so, we can derive an exact relationship between the exchange rate and the difference between the discount factors in each country. Readers are referred to a proof in Backus, Foresi, and Telmer (2001) for a more complete argument. Under the mentioned conditions, and if exchange rate are in units of foreign currency (Euro) per dollar, then

$$\frac{S_{t+1}}{S_t} = \frac{M_{t+1}}{M_{t+1}^*} \tag{5}$$

In logs, we have

$$s_{t+1} - s_t = \Delta s_{t+1} = m_{t+1} - m_{t+1}^* \tag{6}$$

where m_{t+1} and m_{t+1}^* is the domestic and foreign (log) stochastic discount factor, respectively. Taking conditional expectations yields

$$\mathbb{E}_t\left[\Delta s_{t+1}\right] = \mathbb{E}_t\left[m_{t+1}\right] - \mathbb{E}_t\left[m_{t+1}^*\right] \tag{7}$$

In addition, we can also derive an expression for currency risk-premia using the interest rate differential. If interest rates are well-described by only its first two conditional moments, then $y_t^{(1)} = -\log \mathbb{E}_t\left[\exp\left(m_{t+1}\right)\right] = -\mathbb{E}_t\left[m_{t+1}\right] - \frac{1}{2}\sigma_t^2\left[m_{t+1}\right]$. The same is true for foreign interest rates, so $y_t^{(1),*} = -\mathbb{E}_t\left[m_{t+1}^*\right] - \frac{1}{2}\sigma_t^2\left[m_{t+1}^*\right]$. Combining the two expressions relates the interest rate differential to the key moments of the stochastic discount factors:

$$y_t^{(1),*} - y_t^{(1)} = \underbrace{\left(\mathbb{E}_t\left[m_{t+1}\right] - \mathbb{E}_t\left[m_{t+1}^*\right]\right)}_{\text{dollar appreciation}} + \underbrace{\frac{1}{2}\left(\sigma_t^2\left[m_{t+1}\right] - \sigma_t^2\left[m_{t+1}^*\right]\right)}_{\text{risk-premium}} \tag{8}$$

Thus, the currency risk-premia can be expressed as the following expression:

$$\mathbb{E}_t\left[rx_{t+1}^*\right] = y_t^{(1),*} - y_t^{(1)} - \mathbb{E}_t \Delta s_{t+1} \tag{9}$$

As a first pass, we can simply assume that exchange rates simply follow a random walk, thus have no tendency to increase or decrease. Mathematically, this amounts to setting $\mathbb{E}_t \Delta s_{t+1} \approx 0$. Additionally, we can assume that they are a random walk with a drift term. If we build on this instructive assumption, we can simplify the expression for expected currency excess returns to be equal to one-half the difference in the conditional variance of the pricing kernel between the U.S. and Germany:

$$\mathbb{E}_t\left[rx^*_{t+1}\right] = \frac{1}{2}\left(\sigma^2_t\left[m_{t+1}\right] - \sigma^2_t\left[m^*_{t+1}\right]\right) \tag{10}$$

$$\approx y^{(1),*}_t - y^{(1)}_t \tag{11}$$

The key takeaway from these expressions is that the term structure model mathematically relates interest differentials explicitly to the first two conditional moments of the pricing kernels for each country. As one can see from the following expressions, the amount of movement in interest rate differentials is due to either expected changes in exchange rates and/or currency risk-premia. If we assume a small predictable component in exchange rates, it follows from the model that currency risk-premia is simply the interest differential. This is the other benchmark scenario that I consider in the counterfactual tests I conduct. One may also wonder if the truth lies somewhere in between, that there is a predictable component in both currency excess returns and exchange rate changes. Of course, the truth can lie somewhere in between, however the most important detail I wish to scrutinize with regard to the implied currency market prices is the *magnitudes* in the predictable component. This deliberate focus helps us fix ideas about the ability of the term structure model to explain asset market data that it was not designed for. Successful or otherwise, the counterfactual exercise will help us decide if our currency understanding of the data can be improved upon or if the implications are simply too far-fetched. Moving forward, I will outline how both moments are affected by the affine model estimation procedure from these expressions,.

4.1 The Tension of Fitting Interest Rates

How the model trades off fitting conditional means and conditional variances of the pricing kernel will be critical for the exchange rate implications. Cochrane and Piazzesi (2008) model variation in market risk prices as an additional factor, relative to the usual three factors used in term structure models. The model imposes that time variation in the conditional variance of the stochastic discount factor M_{t+1} is *exclusively* driven by the return-forecasting factor. This has implications for how much risk compensation is required to move both bond and currency prices. The large predictable component in bond markets is directly channeled into the conditional variance of the stochastic discount factor through the market price of risk. Because the estimated market risk prices are very sensitive to changes in the return-forecasting factor, time-variation in the conditional mean and the conditional variance of the stochastic discount factor must be equally large. This key point can be made without considerable knowledge of affine term structure models. To quickly summarize the essence of the argument, an affine model relates the conditional mean and variance of the stochastic discount factor to the market price of risk in the following set of equations:

$$-\mathbb{E}_t\left(m_{t+1}\right) = y_t^{(1)} - \frac{1}{2}\kappa\lambda_t^2 \tag{12}$$

$$-\sigma_t^2\left(m_{t+1}\right) = \kappa\lambda_t^2 \tag{13}$$

The amount of captured predictability in bond markets encoded in the return-forecasting factor x_t means that the conditional mean of the log pricing kernel has to fluctuate violently if market risk prices are large, meaning that a great deal of required risk compensation is demanded for relatively modest changes in x_t. Although there is some flexibility in choosing how much risk compensation is required over time via the $\kappa \cdot \lambda_t^2$ term, the model estimation procedure must trade off which conditional moment to emphasize. It turns out that the conditional mean is given priority *at the expense of* the conditional variance of the discount factor. The severity of this trade-off can be further examined in currency market prices. Before I do so, let me outline the affine term structure model.

5 The Affine Term Structure Model

Given the amount of predictability found in bond market data, using the return-forecasting factor may be useful for characterizing price dynamics within a term structure model. Extracting the standard level, slope and curvature factors follows from a principal component decomposition using the residuals after constructing the return-forecasting Factor, and is thus done using forward rates. The comprehensive procedure for extracting the latent factors can be found in Cochrane and Piazzesi (2008). For the purpose of this paper, I give considerable attention to the predictability characteristics the model implies and what that means for currency markets, so I refer interested readers either to the appendix of this paper or Cochrane and Piazzesi (2008) for more specific details.

To ensure that prices satisfy no-arbitrage, we must ensure that the following condition holds

$$P_t^{(n)} = \mathbb{E}_t\left[P_{t+1}^{(n-1)} M_{t+1}\right] \tag{14}$$

In words, this says that a date t bond price with time-to-maturity n is equal to the expected price of the bond appropriately discounted by M_{t+1}. We must also specify how the key risk factors evolve over time. In this model, we impose that the set of factors $\mathbf{Z}_t$ follows a Markovian structure, in particular a first-order VAR,

$$\underbrace{\mathbf{Z}_{t+1}}_{4\text{x}1} = \underbrace{\mu}_{4\text{x}1} + \underbrace{\phi}_{4\text{x}4} \cdot \mathbf{Z}_t + \underbrace{\varepsilon_{t+1}}_{4\text{x}1} \tag{15}$$

where ε_{t+1} is a Gaussian error term with mean zero and variance-covariance matrix $\mathbb{E}\left[\varepsilon_{t+1}\cdot\varepsilon_{t+1}'\right] = \underbrace{\mathbf{V}}_{4\text{x}4}$, and how the market price of risk varies over time, in this case, a linear function of the latent factor,

$$\underbrace{\lambda_t}_{4\text{x}1} = \underbrace{\lambda_0}_{4\text{x}1} + \underbrace{\lambda_1}_{4\text{x}4} \cdot \mathbf{Z}_t,$$

we can express the stochastic discount factor as a function of both the factors and the price of risk,

$$-\log\left(M_{t+1}\right) = \delta_0 + \delta_1' \cdot \mathbf{Z}_t + \frac{1}{2}\lambda_t' \cdot \mathbf{V} \cdot \lambda_t + \lambda_t' \cdot \varepsilon_{t+1},$$

We then posit an exponentially affine functional form for log bond prices, $\log\left(P_t^{(n)}\right) = p_t^{(n)} = A_n + \mathbf{B}_n' \cdot \mathbf{Z}_t$, and solve the no-arbitrage conditions for any point in time t for any maturity n. The result of doing so is equivalent to estimating the cross-sectional loadings $(A_n, \mathbf{B}_n)$ so that the no-arbitrage pricing condition is satisfied. It can be gleaned from the expression for bond prices that for $n = 1$, that $-p_t^{(1)} = y^{(1)} = A_1 + \mathbf{B}_1' \cdot \mathbf{Z}_t = \delta_0 + \delta_1' \cdot \mathbf{Z}_t$ is the one-year nominal interest rate. The remaining parameters are to be estimated using optimization methods where the parameters are chosen to minimize the fit of the data relative to some loss function. For the model under examination, a quadratic loss function was chosen. Alternatively, we could have solved for forward rates since they are a function of prices, $f_t^{(n)} = p_t^{(n)} - p_t^{(n-1)}$. Once we have parameters that well describe these prices, we can derive expressions that precisely characterize the model-implied *expected* bond excess returns.

5.1 Expected Returns and the Market Price of Risk

Linear factor models have analytically tractable expressions that can be particularly useful for both our understanding and further analysis. An equilibrium object of interest is the one-year ahead expected excess return from holding an n-maturity bond, is a function of bond prices as defined previously and is given below:

$$\mathbb{E}_t\left(rx_{t+1}^{(n)}\right) = \mathbf{cov}_t\left(rx_{t+1},\ \underbrace{\varepsilon_{t+1}'}_{\text{shocks to } \mathbf{Z}_t}\right)\cdot\lambda_t - \frac{1}{2}\sigma^2\left(rx_{t+1}^{(n)}\right) \tag{16}$$

In terms of the model parameters, expected excess returns is then

$$\mathbb{E}_t\left(rx_{t+1}^{(n)}\right) = \mathbf{B}_{n-1}'\cdot\mathbf{V}\cdot\lambda_t - \frac{1}{2}\mathbf{B}_{n-1}'\cdot\mathbf{V}\cdot\mathbf{B}_{n-1} \tag{17}$$

Finally, in terms of differences in actual and risk-neutral dynamics and the state variables, expected excess returns is given by the following expression:

$$\mathbb{E}_t\left(rx_{t+1}^{(n)}\right) = \mathbf{B}_{n-1}'\left(\mu-\mu^*\right) + \mathbf{B}_{n-1}'\left(\phi-\phi^*\right)\cdot\mathbf{Z}_t - \frac{1}{2}\mathbf{B}_{n-1}'\mathbf{V}\mathbf{B}_{n-1} \tag{18}$$

We can simplify the expression for expected returns once we adopt the identification restrictions from Cochrane and Piazzesi (2008). That is, if we restrict that (1) expected returns only vary with the x-factor and (2) only level shocks matter for risk-compensation, then risk prices are simply linear in x_t, i.e. $\lambda_t = \lambda_0^{\text{level}} + \lambda_1^{\text{level}}\cdot x_t$. Cochrane and Piazzesi (2008) find that the cross-sectional fit of expected returns along the n time-to-maturity dimension was not affected by restricting returns to only depend on the level shock. I too find evidence documented in the previous section that level shocks play a prominent role in changing expected returns, given that changes in country-specific return-forecasting factors affect returns of all maturities. This is a very useful approximation because it reduces the amount of parameters that need to be estimated. However, given the stylized nature of the model, it is important to recognize how the simplifications made can affect our understanding of the data.

We then have a simple expression for expected excess returns on n-year maturity bonds:

$$\begin{aligned} (19)\ \mathbb{E}_t\left(rx_{t+1}^{(n)}\right) &= \left(B_{n-1}^{\text{level}\prime} cov\left(v_t, \varepsilon_{t+1}^{\text{level}}\right)\lambda_0^{\text{level}} - \frac{1}{2}\mathbf{B}_{n-1}'\mathbf{V}\mathbf{B}_{n-1}\right) + B_{n-1}^{\text{level}\prime} V_{(2,2)}\lambda_1^{\text{level}} \cdot x_t \\ (20) \qquad &\approx \alpha(n) + \beta(n) \cdot x_t \end{aligned}$$

This equation holds for any n-maturity bond, so we can choose any $\lambda = \left(\lambda_0^{\text{level}}, \lambda_1^{\text{level}}\right)$ to match the predictability returns along the cross-section, i.e. for bond returns of maturity n. Cochrane and Piazzesi choose λ so that the slope coefficient is unity and the intercept term is zero. In other words, λ is chosen so that the weighted portfolio of long-maturity bonds will be related to the return-forecasting factor. This is equivalent to a portfolio of bonds weighted by q_r, the dominant eigenvector in the principal component decomposition used to construct the four factors in the term structure model. Implicitly, we are choosing λ so that $\mathbb{E}_t\left(\mathbf{q}_r' \cdot \mathbf{rx}_{t+1}^{(n)}\right) = x_t$. Keep in mind that this portfolio of bonds will have different predictability characteristics than the one that I discuss in section 3 because the choice of a portfolio lets an investor alter their risk exposure, and hence the returns they can achieve. The choice to weight a portfolio of bonds by q_r therefore *maximizes* the fit of expected returns along year-to-maturity n dimension using the return-forecasting factor x_t as a forecasting variable.

Table 4 in the appendix reports the calibrated parameters for a subset of countries where the model behaves well and well-describes the cross-section of bond prices. As I alluded to earlier, the estimated market risk price parameters reveal large magnitude changes. We can then examine how some key assumptions made in the estimation procedure affect these parameter estimates. In particular, restricting that expected returns only move with level shocks, that expected returns only move with the return-forecasting factor, and that the volatility of expected returns are constant over time which is implicitly assumed in the model specification, are essential for identification. As a result, the Jensen's term $\frac{1}{2}\mathbf{B}_{n-1}'\mathbf{V}\mathbf{B}_{n-1} = \sigma^2\left(rx_{t+1}^{(n)}\right)$ depend only on the length of maturity n, but not time t. However, the problem is that the estimation procedure is implicitly trading off fitting the mean and covariance of the yields, but under the mentioned restrictions and model assumption, the fitting procedure favors the conditional first moment. The market price of risk is directly proportional to the conditional variance of the pricing kernel, as can be seen in the following relation

$$\sigma_t^2\left(m_{t+1}\right)=\lambda_t' \cdot \mathbf{V} \cdot \lambda_t \tag{21}$$

which simplifies to the following under the mentioned restrictions:

$$\begin{aligned} \sigma_t^2\left(m_{t+1}\right) &= V_{(2,2)}^2\left(\lambda_0^{\text{level}}+\lambda_1^{\text{level}} \cdot x_t\right)^2 \\ &= \kappa x_t^2 \end{aligned} \tag{22}$$

What remains to be seen is how variation in the market price of risk quantitatively responds to changes in the return-forecasting factor. Choosing λ_1 is essential to the identification of risk-neutral dynamics ϕ^* since the mapping from risk-neutral and actual dynamics is given by the following equation:

$$\phi^*=\phi-\mathbf{V} \cdot \lambda_1 \tag{23}$$

The importance of the conditional homoskedasticity assumption on the variance becomes clear in this expression because it allows the risk adjustment term to be independent of time. It also implies that identification requires large movements in the market price of risk since the relevant entries in $\mathbf{V}$ are small, the entries in $\mathbf{V}^{-1}$ are large so it is necessary for identification that $\lambda_1=\mathbf{V}^{-1}\left(\phi^*-\phi\right)$. In other words, identification of risk-neutral dynamics requires not only that the restriction on how risk compensation varies exclusively with the return-forecasting factor, it also requires time-invariant conditional variance of excess returns.

The framework also links variation in the return-forecasting factor exclusively to the expected excess bond returns and not the conditional variance. However, it could be the case that changes in the return-forecasting factor occur during times of large surprise changes in bond prices. The specification choice renders the model unable to account for the changes in the conditional variance of returns. Although this simplifying approach seems innocuous given that most of the data describes the post-1985 ”Great Moderation era,” where monetary regimes are relatively stable, it remains to be seen what this simplification means for implied asset prices in other markets, and how the risk-return trade-off varies over time. Currency markets and Hansen-Jagannathan bounds can serve as additional avenues for robustness checks that I will employ.

6 Results

I choose to study the U.S. and Germany because these two countries have the longest time series available, with data that span the time period $1/1973 - 5/2009$. Another reason for choosing these two is the in-sample model fit of the U.S. and German bond yields to the affine model are quite good, with root mean squared errors (RMSE's) of less than 10 basis points. I do not report them here in this paper to focus my discussion. More important, the model will capture a large predictable component in their respective bond markets that are consistent with empirical regressions.

To begin testing the implications of the model, first recall the model-implied exchange rates given by the following equation:

$$\mathbb{E}_t^* \left[\Delta s_{t+1} \right] = \left(\mathbb{E}_t \left[m_{t+1} \right] - \mathbb{E}_t \left[m_{t+1}^* \right] \right) \tag{24}$$

The procedure of estimating the affine term structure for two bond markets allows us to construct the conditional mean and variance of the stochastic discount factor, thus enabling us to construct the conditional mean of their bilateral exchange rate change. In simpler terms, we are backing out the predictable component of exchange rates using the affine term structure models for the U.S. and German bond market. From this statistic, we can begin our analysis on the exchange rate implications.

If exchange rates are well-approximated by a random walk, then their conditional mean in its rate of change is approximately zero. Under this scenario, the differences in discount factors are *expected* to cancel out. This scenario does not impose very strong theoretical assumptions, but instead acknowledges the inherently volatile and random nature of exchange rates that is found in the data. Although this benchmark case for currency markets is admittedly strong, it is a useful first pass to ensure that the model-implied exchange rates do not deviate too far from *our current understanding of* the data. We cannot completely dismiss the possibility that exchange rates are more predictable in the data than we have come to believe, but we have to start somewhere. If there is evidence that exchange rate changes are more predictable than a standard random walk benchmark suggests, it can be said that exchange rates are excessively volatile or predictable relative to a meaningful predictor variable.

After estimating the model for the U.S. and Germany, and extracting the implied discount factors from each country, I construct the theoretically implied exchange rate changes with them. I compare them to the data, the model, and the random walk benchmark. To visually illustrate the amount of predictability in exchange rate changes, consider Figure 1. To facilitate comparison, I included a case where the bond risk prices for each country was scaled down by a factor of 2. The case where $\lambda_{(j)}$ is reduced by a factor of 2 is acknowledged by Cochrane and Piazzesi (2008), partly to highlight the fact that there is some estimation uncertainty around this parameter. Even after doing so, the model-implied exchange rate changes deviate substantially from random walk behavior in a rather persistent manner. The results suggest that the size of the predictable component in exchange rates is a lot larger than a random walk specification implies since changes in market prices and hence the return-forecasting factors do not cancel out. This cannot be dismissed particularly if there is additional information in the term structure of interest rates that can be used to improve forecast performance, but it is difficult to accept such a large predictable component without convincing reasons for such a scenario to be plausible.

I also construct the predictable component of currency excess returns, the model-implied currency risk-premia, using the difference in the conditional variance of the pricing kernel between the U.S. and Germany, given by the following expression:

$$\mathbb{E}_t\left[rx^*_{t+1}\right] = \frac{1}{2}\left(\sigma^2_t\left[m_{t+1}\right] - \sigma^2_t\left[m^*_{t+1}\right]\right) \qquad (25)$$

I compare this time-series with the realized excess returns and a *benchmark* for the currency risk premia in the data, namely the interest rate differential, $\mathbb{E}_t\left[rx^{data}_{t+1}\right] \approx y^{(1),*}_t - y^{(1)}_t$. This approximation is consistent with longstanding empirical research on the forward premium anomaly and can be used as a first-pass to understand the exchange rate implications of the model under the assumption that exchange rate changes are unpredictable and thus zero on average. Given the empirical robustness of the forward premium anomaly across a broad set of countries, it seems reasonable to check that the model captures this feature of the data. Figure 2 illustrates the key implication of the model for exchange rate data; there is a surprisingly large predictable component in currency excess returns relative to the benchmark case that is consistent with a forward premium anomaly. It is not immediately clear why differences in the conditional variance of the discount factor implied by the model are not as smooth as interest rate differentials. In any case, the results are surprising and require further analysis for us to make sense of the data. In summary, if we take the findings at face value, I find that the model is inconsistent with our currency understanding of foreign exchange market data.

6.1 Highlighting the Trade-off

To illustrate the consequence of this modeling procedure, we look no further than the magnitude of λ_1^{level}, the key parameter that determines changes in *expected* excess returns with changes in risk. Once again, the linear structure of the model directly links time-variation in the return-forecasting factor to the conditional variance of the pricing kernel. The model further implies that spreads in the return-forecasting factor should *also* be important for expected currency excess returns. The empirical finance literature has extensively documented that interest rate differentials predict currency excess returns. The key question that the model can help us answer is whether the spreads in the return-forecasting factor can forecast currency risk-premia.

Differences in the return-forecasting factor between the U.S. and Germany should correspond to differences in the market price of currency risk and consequently, changes in *expected* currency excess returns. The remaining concern is quantitative in nature; does the model-implied currency risk-premia capture a comparable predictable component with the benchmark case of the interest differential? From Figure 2, it seems that the return-forecasting factor differentials captures a larger predictable component, but empirical regression tests imply that over 94% of the variability in exchange rate changes are *predictable* if we take this model seriously. This is a remarkable amount of predictive power that is more than likely too good to be true. To further test the model, I examine some of its key exchange rate model implications. I take the *expected* exchange rates changes and *expected* excess returns and compare them to each other and to the data. I do this by running empirical regression tests using actual exchange rate data and various predictor variables

$$\Delta s_{t+1} = \alpha + \beta \cdot z_t + \varepsilon_{t+1} \tag{26}$$

where the set of predictor variable z_t consists of the benchmark interest differential and the model-implied currency risk-premia. I also consider the case where bond risk prices are reduced by a factor of 2 and the case where it is reduced by 5. I also include a series of theoretical exercises that relates the model-implied expected exchange rate changes with the same set of predictors. To span the full range of considered outcomes, I also report the cases where the model-implied exchange rate changes are constructed under reduced bond risk prices. Table 3 reports the scenarios under consideration. Interestingly enough, scaling down each country's magnitude risk prices by a factor of 5 delivers produce results that are in line with both the negative regression coefficient estimates found in the data as well as more reasonable explained variability of data in line with the data and benchmark case.

To further illustrate the tension between our current understanding of foreign exchange rates and the documented amount of predictability in bond markets reported earlier in this paper, I illustrate the conflict visually by *simultaneously showing* the time-variability in realized and *expected* excess returns for the financial markets. The first case, Figure 4, displays the scenario that is consistent with the predictable component for U.S. and German government bond markets documented earlier, along with the implied predictable component in their bilateral currency market.

Visually, it is clear that the size of the predictable component in currency markets is a lot larger than forward premium regressions suggest. However, the forward premium empirical tests also seem to suggest that this scenario is highly improbable. The conflict can only be remedied if we reduce the size of predictable component in bond markets. In fact, reducing bond risk prices by a factor of 5 generates a result more consistent with the forward premium in currency markets in the sense that there is a plausible amount of variability explained by term structure relevant predictor variables. Consequently, the

Table 3: Forward Premium Anomaly: Model vs. Data

	Benchmark interest differential $\left(y_t^{(1),*} - y_t^{(1)}\right)$	Model $\left(\mathbb{E}_t\left[rx_{t+1}^*\right]\right)$ *unaltered* risk-price λ	Model $\left(\mathbb{E}_t\left[rx_{t+1}^*\right]\right)$ modified risk-price $\boldsymbol{\lambda/2}$	Model $\left(\mathbb{E}_t\left[rx_{t+1}^*\right]\right)$ modified risk-price $\boldsymbol{\lambda/5}$
	β $t-$stat $R^2(\%)$			
Realized exchange rate Δs_{t+1}	$\mathbf{-0.95}$ (-9.13) 0.17	$\mathbf{0.04}$ (-18.8) 0.00	$\mathbf{0.03}$ (-2.73) 0.00	$\mathbf{0.10}$ (-1.16) 0.00
Model generated exchange rate $(\mathbb{E}_t^*[\Delta s_{t+1}])$ *unaltered* risk-price λ	$\mathbf{-3.45}$ (-1.18) 4.24	$\mathbf{-12.30}$ (-232.4) 99.7	$\mathbf{-0.49}$ (-96.1) 99.7	$\mathbf{-3.94}$ (-82.8) 99.7
Modified model generated exchange rate $(\mathbb{E}_t^*[\Delta s_{t+1}])$ $\boldsymbol{\lambda/2}$	$\mathbf{-0.05}$ (-3.25) 0.07	$\mathbf{-0.06}$ (-329.7) 94.4	$\mathbf{-0.47}$ (-57.2) 94.4	$\mathbf{-1.46}$ (-30.7) 94.4
Modified model generated exchange rate $(\mathbb{E}_t^*[\Delta s_{t+1}])$ $\boldsymbol{\lambda/5}$	$\mathbf{0.83}$ (-4.06) 61.9	$\mathbf{-0.01}$ (-157.6) 15.1	$\mathbf{-0.10}$ (-21.3) 15.1	$\mathbf{-0.30}$ (-8.10) 15.1

Note: I conduct the following empirical tests involving data and theory under the following scenarios: I first run the standard forward premium regression test by regressing exchange rate changes on interest differentials. I then regress realized exchange rate changes on the model-implied *predictable* currency excess return. I include the case in which the risk price parameters are reduced by half and by a factor of 5. I also include an auxiliary set of theoretical exercises where I relate the model generated exchange rate changes on interest differentials as well as the model implied currency risk-premia. The reported set of statistics for each experiment (top to bottom) include the regression coefficient, the t-statistic and the R^2. The t-statistics are associated with the null hypothesis that uncovered interest parity holds.

predictable component in each bond market is reduced to nothing. Figure 4 visually demonstrates this point, one of the main results in this paper.

7 Discussion

7.1 Modeling Tradeoff

The exchange rate implications of the four-factor affine term structure model of Cochrane and Piazzesi (2008) reveal an inherent modeling trade-off. Based on our current understanding, it is either the case that we accept that there is predictability in two separate bond markets and way more predictable variation in currency prices than commonly documented. Alternatively, if we accept our current understanding of currency markets, in the existence of predictable returns that is consistent with empirical regressions in terms of amount of variability explained in the time-series and the failure of uncovered interest parity, then according to the counterfactual estimates, there is very little predictability in bond markets. This is certainly puzzling because in the data there is meaningful and statistically significant evidence of predictable bond excess returns internationally. The critical limitation I intend to highlight is that there cannot be middle ground; the tension of simultaneously explaining predictable returns in bond and currency prices cannot be resolved.

The limitation is related to this class of linear factor models that we use to understand the data. The implicit assumptions embedded into the estimation procedure emphasize the fit of conditional means over conditional covariances. As a result, the stochastic properties of the pricing kernel that is paramount to our understanding of the data belies a large amount of anticipated and exploitable predictability. Despite its shortcomings, the four-factor linear affine model offers important lessons about our current understanding of bond and currency return predictability. For one, it is probably the case that more than random walk shocks are priced. This is not surprising if conditional Sharpe ratios are consistent with researchers' reasonability criteria (Duffee (2010)). With that in mind, it is important to mention that there is broad empirical support of this identification assumption not just in the U.S. data but also for a broad set of industrialized government bond markets. Unfortunately, the identification restriction is not an innocuous one. The puzzling implications only become apparent when we extrapolate to currency markets. In any case, the results and limitations I have discuss lead to the central result of this paper:

Proposition 1 *Within a four-factor affine term structure model of Cochrane and Piazzesi (2008) with time-invariant shocks to the underlying state variable (conditionally homoskedastic returns), given that variation in risk-premia move exclusively with random walk shocks, if there is predictability in two individual bond markets, then there is a counterfactually large amount of predictability in exchange rates and currency risk-premia. Conversely, if the market risk prices are consistent with our current understanding of currency market data, then expected excess returns in bond markets are nearly unpredictable.*

The intuition for this result can be gleaned from the relationship between the risk-neutral dynamics and actual dynamics via the market risk price. Risk-neutral dynamics are equal to the actual dynamics, less an adjustment for risk, i.e. $\phi^{*}= \phi - \mathbf{V}\lambda$. Reducing the magnitude change in risk prices means that the parameter λ_1^{level} must be scaled down. The result is a much smaller risk adjustment in the risk-neutral dynamics, so that it is indistinguishable from actual dynamics, i.e. $\phi^{*} \approx \phi$. The bond risk-premia for holding onto an n-year bond is then $\mathbb{E}_t\left(rx_{t+1}^{(n)}\right) \approx \mathbf{B}'_{n-1}\left(\phi - \phi^{*}\right) \cdot \mathbf{Z}_t \approx 0$ making it zero in expectation.

7.2 An Uncertainty Principle in Bond Markets

Linear factor models can quickly remedy the modeling trade-off by admitting time-varying volatility. This modeling *choice* can simultaneously resolve some of the issues that have been raised in this paper. For one, it makes the dynamic risk-return trade-off more plausible with reasonable restrictions discussed in detail by Duffee (2010). The underlying tension in characterizing the risk-return trade-off in bond markets is apparent in how the Sharpe ratio varies over time. Fortunately for linear factor models, the link between predictable returns and conditional Sharpe ratios is made explicit through market risk prices. For a more detailed derivation of conditional Sharpe ratios, I refer you to Duffee (2010), but for the purpose of my discussion, I examine the stochastic properties of the conditional Sharpe ratios for log returns. The expression for the Sharpe ratio for log returns is then given by:

$$\Theta_t = \frac{\mathbb{E}_t\left[r_{i,t+1}\right] - r_{t+1}^{f} + \frac{1}{2}\sigma_t^2\left[r_{i,t+1}\right]}{\sigma_t\left[r_{i,t+1}\right]} \tag{27}$$

Under some conditions, we can replace $R_{i,t+1}$ with the excess return, RX_{t+1}. [2]

In doing so I obtain the following expression for the conditional variance of the pricing kernel, which for bond markets is the maximal Sharpe ratios that can be achieved, given

[2] We can take a log-linear approximation of excess returns around $\log\left(RX_{t+1}\right) = 0$. We also rule out short positions on a portfolio of bonds that is typically used to maximize Sharpe ratios. The portfolio we consider gives positive weights to each of the n-year-to-maturity bonds. Overall, these conditions are restrictive but it is not unreasonable to expect them to be satisfied.

below:

$$\sigma_t\left[m_{t+1}\right]=\frac{\mathbb{E}_t\left[rx_{t+1}\right]+\frac{1}{2}\sigma_t^2\left(rx_{t+1}\right)}{\sigma_t\left[rx_{t+1}\right]} \tag{28}$$

The estimated bond risk price magnitudes for a set of countries where Cochrane and Piazzesi's (2008) model performs well are included in the appendix. To focus the discussion, let us consider carefully the magnitude of these estimates and what they mean for how *expected* excess returns are modeled to vary over time. What is implied by the equation above is that conditional Sharpe ratios are just as volatile as *expected* excess returns in bond markets. According to my results, the heightened variability is inherited by the implied movements in the *expected* excess returns in currency markets. Admitting stochastic volatility in an otherwise standard linear factor model would reduce the amount of variability channeled into market risk prices and hence would reduce the amount of variability in conditional Sharpe ratios since movements in the conditional volatility of excess returns will offset movement in expected excess returns.

Another possibility to consider is that the amount of movement in expected returns is an artifact of the data, not nearly moving around as much it is measured to do. This is somewhat difficult to defend in light of my findings. Evidence of statistical predictability in bond markets is robust and hard to ignore. Moreover, if we decide that our current understanding of the data requires modification, it is also plausible that the truth lies somewhere in between. In other words, because bond excess returns are predictable for a broad set of industrialized economies' bond markets, there must be a convincing explanation for why the size of the predictable component in bond markets is overstated. These are complicated details to sort out and for many situations, these findings may provide little value.

If one chooses to remain agnostic about the matter because jointly accounting for the statistical evidence of predictable bond and excess returns is not important to the research question, one can simply assume that the volatility of bond returns is time-varying. By assuming conditionally heteroskedastic returns, the one-to-one relationship between market risk prices and the stochastic discount factor is uncoupled. This specification is consistent with a reduction of the amount of plausibly anticipated predictable returns in bond markets concluded by Duffee (2010) who constrains the Sharpe ratio to not exceed "reasonable" bounds. In fact, breaking the direct link between predictable excess returns and risk is precisely what we do when we admit time-varying volatility of returns in linear factor models.

Although the specification choice of time-varying volatility can shore up many issues regarding our joint understanding of predictable bond and currency returns, this choice comes with its own set of trade-offs. The consequence of this choice reduces the usefulness of linear factor models for real-time forecasting because states of the world of high realized excess returns tend to occur during times of greater return variability. As I discussed in the previous section, for the Hansen-Jagannathan bounds to be consistent with both bond and currency market data, it must be the case that expected excess returns be conditionally heteroskedastic to offset movements in expected excess returns. However, if we look at the relation between the return-forecasting factor and the conditional Sharpe ratios given below:

$$
\begin{aligned}
\left(V_{(2,2)}\right)^{\frac{1}{2}} \cdot \left(\lambda_0^{\text{level}} + \lambda_1^{\text{level}} \cdot x_t\right) &= \\
\sigma_t\left[m_{t+1}\right] &= \frac{\mathbb{E}_t\left[rx_{t+1}\right] + \frac{1}{2}\sigma^2\left(rx_{t+1}\right)}{\sigma\left(rx_{t+1}\right)}
\end{aligned} \tag{29}
$$

we notice that conditionally heteroskedastic returns means two terms are moving on the right hand side. If so, then variation in the return-forecasting factor signal changes to either the conditional mean or the conditional variance of risk-premia, thereby reducing the amount of exploitable predictability implicit in the return-forecasting factor.

Proposition 2 *Given the mentioned restrictions on the market price of risk, variation in the return-forecasting factor cannot be exclusively attributed to expected excess returns and can also precede movement in the conditional volatility of returns. Moreover, the amount of measured predictability in bond excess returns captured in the return-forecasting factor of Cochrane and Piazzesi* (2005, 2008) *is too large to be consistent with exploitable predictability in bond excess returns.*

In light of the highlighted limitations, it is also important to recognize some alternative possibilities that can explain some of the persistence in the bond excess return data. As Hamilton and Wu (2010) point out, it could be that the specification choice of linear affine models is inconsistent with the data and that there are serially correlated errors across maturity that can show up as predictable returns. It is also possible that variation in bond risk-premia captured by the return-forecasting factor is picking up *ex-post* information that leads one to believe there is much more predictable returns when viewed through the rear-view mirror than an amount that is perceived to be advantageous in real-time. Whatever the case may be, the return-forecasting factor is constructed to package information that, in a statistical sense, is *measured* to track realized excess returns and how it changes over time, but likely overstates the amount of movement in *expected* excess returns. Although it is unclear why the return-forecasting factor captures so much variability in excess returns in the data, I argue that the reason for its overstatement of exploitable predictability stems from the estimation procedure that implicitly fixes the volatility of bond excess returns to be constant, a choice that can only be made if it is *assumed that an econometrician has access* to all historical data.

These results shed more light on the implicit trade-off that underly the estimation of linear affine models. Because of the emphasis on fitting conditional means limits its usefulness for understanding time-variation in predictable returns, linear factor models cannot adequately capture changes in the conditional variance of excess returns. Further, if the shocks to the state variable are conditionally heteroskedastic, then this risk-adjustment term is time dependent and thus, time-variation in market risk prices cannot directly be linked to the return-forecasting factor. Further, dynamic Sharpe ratios cannot uniquely be determined to move with the conditional mean of excess returns or the conditional variance of returns. In all, the results highlight an auxiliary uncertainty principle that limits the usefulness of the return-forecasting factor for real-time forecasting.

Corollary 3 *Given the mentioned restrictions on the market price of risk, if there is time-varying volatility, then one cannot track expected returns on bonds without accounting for changes in the conditional variance of excess returns.*

7.3 Some Commentary

I highlight limitations of the class of affine term structure models in its ability to simultaneously capture the predictable returns in bond and currency markets. More broadly, I assess the ability of linear factor models in helping with our understanding of the stochastic properties of excess returns in distinct but related markets. In doing so, I demonstrate that there are useful judgment calls that need to be made that are not entirely innocuous. Of course, modeling frameworks are not without their limitations but it is clear at least to myself that clearly outlining these limitations brings about a deeper understanding of the ideas and forces with which we hope to capture empirical reality. Highlighting the limitations of our models makes it very transparent that our understanding of phenomena are constrained by the model we choose to employ and that different modeling approaches may have different strengths and weaknesses. Nonetheless, their respective importance hinges upon the key insights that can be extracted from them. After all, models are only approximations of reality. The important thing to realize is that they give us an understanding of certain aspects of the data and the world that is not possible with empirics alone.

Using a linear class of models to jointly account for time-varying risk and return in bond and currency markets is, as I have shown, asking too much from the model. An alternative that does provide a joint account of time-varying risk in bond and currency markets is Alvarez, Atkeson, and Kehoe (2009). Directly linking a stochastic discount factor to key economic risk factors in a general equilibrium model is important for our understanding of time-varying risk, but their framework choice requires that they be agnostic about the amount of measured predictability in bond markets. This is neither right nor wrong, but a direct consequence of their modeling choice. Nonetheless, it still may be useful for understanding key features of the data.

At the other end of the spectrum, if one does want to tell a risk-based story of predictable returns in financial markets, then one must account for a large amount of predictability in bond markets as Cochrane and Piazzesi (2005, 2008) find, and in currency markets recently documented by Lustig et. al (2010). Doing so with economic interpretability through key risk factors in each respective market in relation to macroeconomic aggregates would be enormously useful as a guide for thinking and intuition, one that many would welcome.

There are of course other possible explanations that are consistent with empirical reality. One possibility is that the amount of statistically measured predictability is not only smaller than the data seem to capture, but that it may not exclusively be driven by time-varying risk factors. An avenue that has not been discussed very extensively in the literature is the importance of the *subjective beliefs of market participants* and how they can reduce the amount of measured predictability that a stochastic discount factor must explain. To show how subjective beliefs can matter, one can decompose measured predictability of returns according to the following expression:

$$\widehat{\mathbb{E}}\left[R^e_{t+1}\right] = \underbrace{\left(\widehat{\mathbb{E}}_t\left[R^e_{t+1}\right] - \tilde{\mathbb{E}}_t\left[R^e_{t+1}\right]\right)}_{(*)} - \frac{\widetilde{\mathbb{C}ov}_t\left(M_{t+1}, R^e_{t+1}\right)}{\mathbb{E}_t\left[M_{t+1}\right]} \tag{30}$$

where $\widehat{\mathbb{E}}$ is expectations from the actual statistical probability distributions and $\tilde{\mathbb{E}}$ is taken under the subjective beliefs of the market participants as is the covariance operator $\widetilde{\mathbb{C}ov}_t(\cdot)$. The cost of taking this approach is the abandonment of a commonly made auxiliary assumption of rational expectations made to estimate the parameters of a model. Assuming rational expectations amounts to setting the $(*)$ term to zero, consequently demanding more explanatory variation in the candidate risk factors that make up the pricing kernel in order for them to exclusively drive variation expected excess returns. The results in this paper indicate that this may be asking too much from the discount factor. Notable research has attempted to break the one-to-one link between expected excess returns and risk factors, namely Piazzesi and Schneider (2011) and Gourinchas and Tornell (2004). One plausible motivating reason for why agents make systematic forecast errors is because they solve a signal extraction problem, trying to figure out the best ways to distinguish transitory disturbances from permanent shocks. The inability of a forecasting agent to decompose short-lived and long-lived disturbances means that they are necessarily making forecasts that minimize how wrong they are and these forecast mistakes show up as measured predictability. The full implications of such a modeling approach can help us explain time-series properties of returns not only for bond markets as Piazzesi and Schneider (2011) show, but for any financial market, making it an interest avenue for future research.

8 Conclusion and Future Directions

After documenting that predictable returns in bond markets is found in other industrialized economies, I determine if the foreign exchange market prices implied by the term structure model of Cochrane and Piazzesi (2008) are consistent with our current understanding of predictable currency risk-premia. I find that currency market prices are substantially more predictable than standard predictor variables suggest. According to my results, even if we wanted to jointly account for the predictable component in bond and currency risk-premia, we could not; linear factor models can account for either bond or currency return predictability, but not both markets simultaneously. This limitation found in the class of linear models is in large part due to the emphasis on fitting the conditional mean of the pricing kernel, among other things.

The results in this paper have implications for the asset prices and dynamic economic models as well. Existing frameworks such as an affine term structure model and (New) Keynesian models that form the basis of policy rules generate movement in the conditional mean of the pricing kernel. At business cycle frequencies, changes in the policy instrument, the short-term interest rate, do not correspond to changes in the conditional mean but the conditional variance of the pricing kernel. This result is well-documented in finance and was recently emphasized by Alvarez, Atkeson and Kehoe (2007). By not accounting for changes in risk, their policy response may have unintended consequences. This criticism is especially relevant for rules that may have worked in the past but may have also outlived their usefulness. One immediate implication of my findings is that inflation expectations management using a Taylor rule alters risk in financial markets. If this is not the intention of central banks, is there a better way to manage expectations?

The topic of time-varying bond and currency excess returns cannot be fully addressed then without considering the role of the central bank. Through the short-term interest rate over the past several decades and recently the long-term interest rate, central bank policy plays a critical role not only in changing financial risk conditions but also in coordinating future economic activity. Because of their importance, key questions remain unanswered. Does monetary policy cause changes in risk, or is time-variation in risk-premia driven by deeper forces that are outside the direct control of central bank officials? These questions are especially important in light of recent findings that financial market data often move together across countries, suggesting that global forces are at work. In light of recent events, we can benefit from understanding alternative methods with which policy makers can manage risk and maintain financial market stability. These concerns should be, if they are not already, research priorities in monetary economics.

In my future research, I hope to further our understanding of why returns in financial markets move so much. For bond markets, this requires a better understanding of how expectations are formed by financial market participants. Jointly capturing bond and currency market prices require careful treatment of expectations formation that takes seriously financial market participants' concern for structural change. These considerations may yield interesting avenues with which we can uncover an integrated understanding of both bond and currency market prices. I hope this paper provides an impetus for such an undertaking, as well as tantalizing leads for future research.

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9 Appendix

9.1 Data Description

Data on zero-coupon bond yields are taken from Jonathan H. Wright, further details on the construction of the data using Svensson, Spline, or Nelson-Siegel approaches to extracting zero-coupon yields are found on www.econ2.jhu.edu/People/Wright/intyields.pdf.

9.2 Solving for Bond Market Prices

To begin, we first conjecture an exponentially affine function form of the bond price given by the following

$$P_t^{(n)} = \exp\left(A_n + \mathbf{B}'_n \cdot \mathbf{Z}_t\right) \tag{31}$$

and in logs, we get:

$$\begin{aligned} p_t^{(n)} &= A_n + \mathbf{B}'_n \cdot \mathbf{Z}_t & (32) \\ y_t^{(n)} &= -\frac{1}{n} p_t^{(n)} & (33) \\ &= -\frac{A_n}{n} - \frac{1}{n} \mathbf{B}'_n \cdot \mathbf{Z}_t \end{aligned}$$

Written in terms of the stochastic discount factor, bond prices are then

$$P_t^{(n)} = \mathbb{E}_t\left[M_{t+1} P_{t+1}^{(n-1)}\right] \tag{34}$$

Suppose that the evolution of the state variable can be characterized as a first-order VAR:

$$\mathbf{Z}_{t+1} = \mu + \phi \cdot \mathbf{Z}_t + \varepsilon_{t+1} \tag{35}$$

where ε_{t+1} is a multivariate normal distribution with mean $\vec{0}$ and variance-covariance matrix $\mathbf{E}\left[\varepsilon_{t+1}\varepsilon'_{t+1}\right] = \mathbf{V}$. Also, suppose the (log) discount rate m_{t+1} is an affine function of the state variable along with an additional stochastic term magnified by the market price of risk,

$$m_{t+1} = -\delta_0 - \delta'_1 \cdot \mathbf{Z}_t - \frac{1}{2}\lambda'_t \cdot \mathbf{V} \cdot \lambda_t - \lambda'_t \cdot \varepsilon_{t+1} \tag{36}$$

where the market price of risk is also affine in the state variable $\lambda_t = \lambda_0 + \lambda_1 \cdot \mathbf{Z}_t$. Given that shocks to the state variable are conditionally homoskedastic, the conversion of actual dynamics to risk-neutral dynamics involves an adjustment for risk, given by the following relation:

$$\mu^* = \mu - \mathbf{V} \cdot \lambda_0 \tag{37}$$

$$\phi^* = \phi - \mathbf{V} \cdot \lambda_1 \tag{38}$$

To solve for no-arbitrage prices, we must ensure that the loadings $(A_n, \mathbf{B}_n)_{n=1}^N$ satisfy the equilibrium prices for any t and for all n. We proceed by induction. To ensure that $p_t^{(0)} = 0$, it must be that $A_0 = 0, \mathbf{B}_0 = \tilde{\mathbf{0}}$. Now, let us solve for the one-year bond price. Then, I assume that the condition holds for some maturity n and then show that it holds for $n+1$. To start for $n = 1$,

$$P_t^{(1)} = \exp\left(A_1 + \mathbf{B}_1' \cdot \mathbf{Z}_t\right)$$

The one-year price should be equal to negative the one-year interest rate, which is given by the following expression:

$$\begin{aligned} y_t^{(1)} &= \log \mathbb{E}_t\left[\exp\left(m_{t+1}\right)\right] \\ &= \log\left(\exp\left(\mathbb{E}_t m_{t+1} + \frac{1}{2}\mathbb{V}_t m_{t+1}\right)\right) \\ &= \left(-\delta_0 - \delta_1' \cdot \mathbf{Z}_t\right) \\ &= -p_t^{(1)} = A_1 + \mathbf{B}_1' \cdot \mathbf{Z}_t \end{aligned}$$

which is true if $\delta_0 = -A_1$ and $\delta_1 = -\mathbf{B}_1$. Now, suppose that $P_t^{(n)} = \exp\left(A_n + \mathbf{B}_n' \cdot \mathbf{Z}_t\right)$ and show that $P_t^{(n+1)} = \exp\left(A_{n+1} + \mathbf{B}_{n+1}' \cdot \mathbf{Z}_t\right)$ by explicitly solving for $(A_{n+1}, \mathbf{B}_{n+1})$. To do so,

$$
\begin{aligned}
(39)\qquad P_t^{(n+1)} &= \mathbb{E}_t\left[\exp\left(m_{t+1}+p_{t+1}^{(n)}\right)\right] \\
(40)\qquad &= \mathbb{E}_t\left[\exp\left(\begin{array}{c} -\delta_0-\delta_1'\cdot\mathbf{Z}_t-\frac{1}{2}\lambda_t'\cdot\mathbf{V}\cdot\lambda_t+\lambda_t'\cdot\varepsilon_{t+1} \\ +A_n+\mathbf{B}_n'\cdot\mathbf{Z}_{t+1}\end{array}\right)\right] \\
(41)\qquad &= \exp\left(-\delta_0+A_n\right)\quad \underbrace{\mathbb{E}_t\left[\exp\left(B_n'\cdot Z_{t+1}\right)\right]}_{\exp\left(\mathbf{B}_n'(\mu^*+\phi^*\mathbf{Z}_t)+\frac{1}{2}\mathbf{B}_n'\cdot\mathbf{V}\cdot\mathbf{B}_n\right)} \\
(42)\qquad &= \left[\exp\left(\underbrace{-\delta_0+A_n+\mathbf{B}_n'\cdot\mu^*+\frac{1}{2}\mathbf{B}_n'\cdot\mathbf{V}\cdot\mathbf{B}_n}_{A_{n+1}}+\underbrace{\left(\delta_1'+\mathbf{B}_n'\cdot\phi^*\right)}_{\mathbf{B}_{n+1}'}\cdot Z_t\right)\right] \\
(43)\qquad &= \exp\left(A_{n+1}+\mathbf{B}_{n+1}'\cdot\mathbf{Z}_t\right)
\end{aligned}
$$

where expectations are taken under the risk-neutral probability distribution. Thus, the expression for the loading coefficients given above satisfies the equilibrium condition. Solving for forward rates is straightforward from the following relation

$$
\begin{aligned}
(44)\qquad f_t^{(n-1\to n)} &= p_t^{(n-1)}-p_t^{(n)} \\
&= A_{n-1}-A_n+\left(\mathbf{B}_{n-1}-\mathbf{B}_n\right)'\cdot\mathbf{Z}_t \\
(45)\qquad &= A_n^f+\mathbf{B}_n^{f\prime}\cdot\mathbf{Z}_t
\end{aligned}
$$

where $A_n^f = A_{n-1}-A_n = \delta_0-\mathbf{B}_{n-1}'\cdot\mu^*-\frac{1}{2}\mathbf{B}_{n-1}'\cdot\mathbf{V}\cdot\mathbf{B}_{n-1}$ and

$$
\begin{aligned}
(46)\qquad \mathbf{B}_n^f &= \mathbf{B}_{n-1}-\mathbf{B}_n=\delta_1'+\mathbf{B}_{n-2}'\phi^*-\delta_1'-\mathbf{B}_{n-1}'\phi^* \\
&= \left(\mathbf{B}_{n-2}-\mathbf{B}_{n-1}\right)'\phi^* \\
(47)\qquad &= \left(\mathbf{B}_{n-3}-\mathbf{B}_{n-2}\right)'\phi^{*2} \\
\ldots &= \left(\mathbf{B}_0-\mathbf{B}_1\right)'\cdot\phi^{*n-1} \\
(48)\qquad &= \delta_1'\phi^{*(n-1)}
\end{aligned}
$$

Numerical techniques that solve for $\left(A_n^f,\mathbf{B}_n^f\right)_{n=1}^N$ used in Cochrane and Piazzesi (2008) compare the fit of the model to the data with the best linear estimator of forward interest rates using nonlinear optimization techniques. Not only that, the fit of the model is assessed using a quadratic loss function, which is sensible given that shocks to the under-

Table 4: Estimated Bond Risk Prices

λ	λ_0^{level}	λ_1^{level}
US	-1.28	-143
DE	-1.45	-215
JP	-1.37	-441
AUS	-2.13	-217

Note: Market risk price estimates for a subset of international countries

lying state variable are assumed to be Gaussian with time-invariant volatility.

9.3 Cross-Country Estimates of the Market Price of Risk

The cross-sectional fitting procedure is critical for the estimates of the market price of risk. The procedure is made even simpler once we only allow random walk shocks to move risk-premia. Given this restriction, we can write the market price of risk to be a function of only the return forecasting factor x_t,

$$\lambda_t = \lambda_0^{\text{level}} + \lambda_1^{\text{level}} \cdot x_t, \tag{49}$$

making the key parameters that govern the time-series properties of the market price of risk only two dimensional instead of twenty if all of the four factors affected the price of risk. For a set of countries where the affine model performs relatively well, I report the estimates of $\lambda = \left(\lambda_0^{\text{level}}, \lambda_1^{\text{level}}\right)$ given below in Table 2:

There are clearly idiosyncratic differences in the parameter estimates that can be attributed to country specific reasons as well as when the bond data become available. However, a common pattern for this set of countries is the sheer magnitude of the estimates for λ_1^{level}. To be sure, the size of λ_1^{level} determines how sensitive a change in risk compensation is to the return-forecasting factor. Based on the large magnitudes in the estimate of $\lambda_{1,l}$, small changes in x_t set off large movements in risk-premia for a set of countries. The interpretation of this parameter estimate is that a 100 basis point increase in the return forecasting factor generates a $143\ (.01 * 143\,(.01) = 1.43)$ basis point increase in the amount of compensation required for buying a short-maturity bond.

9.4 Sharpe Ratios

We can derive Sharpe ratios for simple returns using the basic asset pricing moment condition

$$\mathbb{E}_t\left[M_{t+1}R^e_{t+1}\right] = 0 \tag{50}$$

We can expand this expression using some key statistical relations:

$$\begin{aligned} \mathbb{E}_t\left[M_{t+1}R^e_{t+1}\right] &= \mathbb{E}_t\left[M_{t+1}\right]\mathbb{E}_t\left[R^e_{t+1}\right] + \mathbb{C}ov_t\left(M_{t+1}, R^e_{t+1}\right) \\ &= \mathbb{E}_t\left[M_{t+1}\right]\mathbb{E}_t\left[R^e_{t+1}\right] + \mathbb{C}orr_t\left(M_{t+1}, R^e_{t+1}\right)\sigma_t\left(M_{t+1}\right)\sigma_t\left(R^e_{t+1}\right) \end{aligned} \tag{51}$$

Rearranging, we get

$$\frac{\sigma_t\left(M_{t+1}\right)}{\mathbb{E}_t\left[M_{t+1}\right]} = -\frac{\mathbb{E}_t\left[R^e_{t+1}\right]}{\sigma_t\left(R^e_{t+1}\right)}\frac{1}{\mathbb{C}orr_t\left(M_{t+1}, R^e_{t+1}\right)} \tag{52}$$

Since $\mathbb{C}orr_t\left(M_{t+1}, R^e_{t+1}\right) \in [-1, 1]$, we can simplify the expression:

$$\frac{\sigma_t\left(M_{t+1}\right)}{\mathbb{E}_t\left[M_{t+1}\right]} \geq \frac{\mathbb{E}_t\left[R^e_{t+1}\right]}{\sigma_t\left(R^e_{t+1}\right)} \tag{53}$$

Notice that if we take the stochastic discount factor to be perfectly negatively correlated with expected excess returns, then the expression holds with equality.

For any stochastic return process, these bounds hold. We can then construct Hansen-Jagannathan bounds to understand the properties of the implied stochastic discount factor to see how much our model can explain the variability of expected excess returns.

For log-linear models, we can write down the log Sharpe ratios

$$s^i_t = -\mathbb{C}orr_t\left(m_{t+1}, rx_{t+1}\right)\sigma_t\left(m_{t+1}\right) \tag{54}$$

where lowercase variables are in logs. We also know what the (log) risk-free rate is

$$
\begin{aligned}
r_{f,t}^{(1)} &= -\mathbb{E}_t\left[\exp\left(m_{t+1}\right)\right] \\
&= -\left(\mathbb{E}_t\left[m_{t+1}\right]+\frac{1}{2}\sigma_t^2\left[m_{t+1}\right]\right) \qquad (55)
\end{aligned}
$$

For a strictly positive cum dividend value process, the log conditional Sharpe ratio can be written as

$$
\begin{aligned}
s_t^i &= \frac{\mathbb{E}_t\left[r_{i,t+1}\right]-r_{f,t}+\frac{1}{2}\sigma_t^2\left(r_{i,t+1}\right)}{\sigma_t\left(r_{i,t+1}\right)} \qquad (56) \\
&\approx \frac{\mathbb{E}_t\left[rx_{t+1}\right]+\frac{1}{2}\sigma_t^2\left(rx_{t+1}\right)}{\sigma_t\left(rx_{t+1}\right)} \qquad (57)
\end{aligned}
$$

for any (log) return process $r_{i,t+1}$. The approximation holds for (57) for values of rx_{t+1} around zero. For now, let us suppose that the unconditional mean of rx_{t+1} is approximately zero. This seems to be a reasonable approximation.

For the restrictions on the market price of risk in Cochrane and Piazzesi (2008) the choice of $\left(\lambda_0^{\text{level}},\lambda_1^{\text{level}}\right)=\lambda$ is equivalent to fitting the cross-sectional regression linking both an unconditional mean expected excess return and the time-variation of expected returns exclusively through movements in the return-forecasting factor (only in relation to random walk shocks). In addition, when the portfolio of bonds is weighted by $\mathbf{q}_r'$, the magnitude change in expected returns as a function of x_t implies wild swings in the conditional Sharpe ratio, which in turn is linearly related to the conditional volatility of the pricing kernel for the class of linear affine models. We can see this with the following linear-affine relation between the Sharpe ratios and expected excess returns:

$$
\begin{aligned}
\sigma_t\left(m_{t+1}\right) &= s_t^{(1)}=\frac{\kappa_0+\kappa_1\mathbb{E}_t\left[rx_{t+1}\right]}{\sigma^2\left(rx_{t+1}\right)} \qquad (58) \\
&= \kappa_0\left(\sigma^2\left(rx_{t+1}\right)^{-1}\right)+\kappa_1 a\left(1\right)+\kappa_1\beta\left(1\right)\sigma^2\left(rx_{t+1}\right)^{-1}\cdot x_t \qquad (59) \\
&= \gamma_0+\gamma_1\cdot x_t \qquad (60)
\end{aligned}
$$

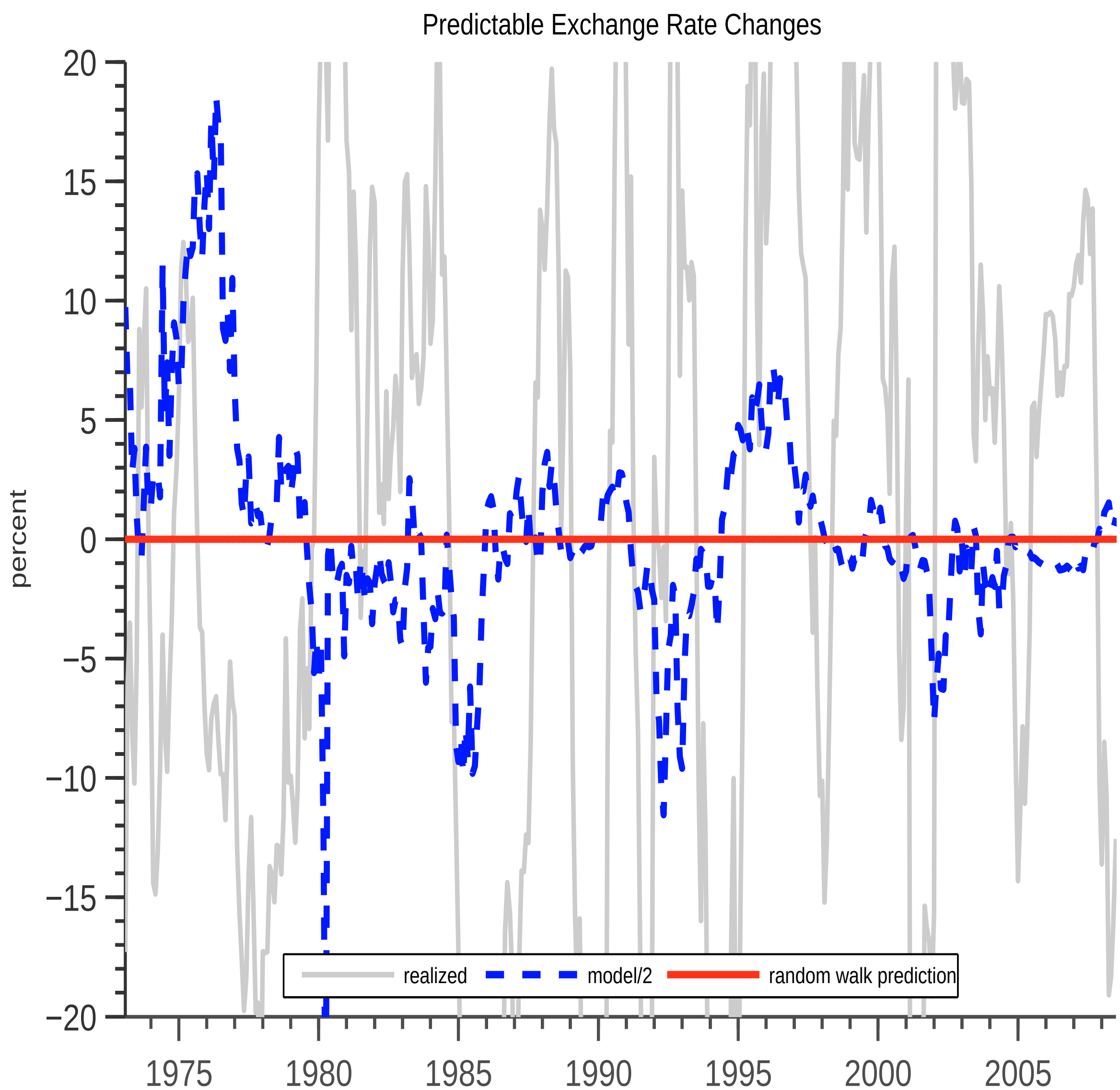
where $\left(\kappa_0,\kappa_1,\gamma_0,\gamma_1\right)$ are arbitrary constants, and $\mathbb{E}_t\left[rx_{t+1}\right]=a\left(1\right)+\beta\left(1\right)\cdot x_t$ where x_t is the return-forecasting factor constructing from average term-premia not yet weighted by any portfolio. In general, we have the following affine relation between expected returns to the return-forecasting factor $\mathbb{E}\left[rx_{t+1}^{(n)}\right] = a\left(n\right) + \beta\left(n\right) \cdot x_t$. Once we weight the portfolio of bonds by $\mathbf{q}_r'$, we get the cross-sectional average of returns is given by $\mathbb{E}_t\left[\mathbf{q}_r' \cdot \mathbf{rx}_{t+1}^{(n)}\right] = \tilde{x}_t$, where the $\widetilde{(\cdot)}$ denotes variables that are weighted by the portfolio $\mathbf{q}_r$. We can then relate the conditional volatility of the pricing kernel $\sigma_t\left(m_{t+1}\right)$ to the weighted return-forecasting factor with the following weighted regression

$$
\begin{aligned}
\sigma_t\left(m_{t+1}\right) &= \frac{1}{2} a\left(n\right) \sigma\left(rx_{t+1}^{(n)}\right) + \underbrace{\beta\left(n\right)\left[\sigma\left(rx_{t+1}^{(n)}\right)\right]^{-1}}_{\text{weighted regression coefficient}} \cdot x_t \\
&= \tilde{a}\left(n\right) + \theta \cdot x_t \qquad (61)
\end{aligned}
$$

so for any n time-to-maturity, $\beta\left(n\right)\left[\sigma^2\left(rx_{t+1}^{(n)}\right)\right]^{-1} = \theta$, a constant that needs to be large to match the large amount of predictability at long horizons. For the U.S. the weighted regression coefficient is $\hat{\theta} = 10.67$, which implies large magnitude changes in the term $\sigma_t\left(m_{t+1}\right)$ associated with variation in x_t.

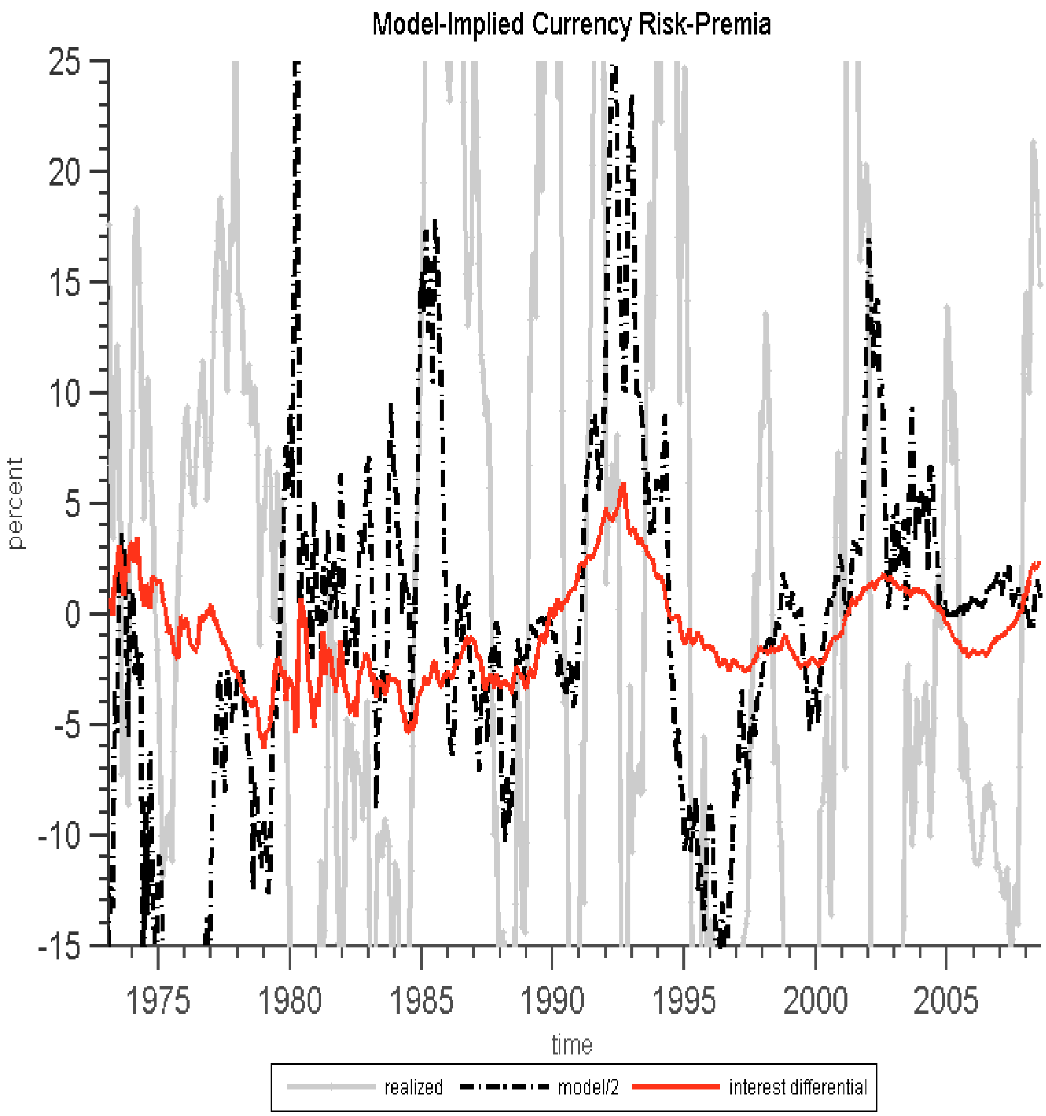
The identification restriction allows the model to explain most of the cross-sectional variation in expected returns, but it implies a conditional Sharpe ratio that increases linearly with n years-to-maturity. The problem with this result is that Duffee (2010) argues and provides evidence that Sharpe ratios are inversely related to maturity. Intuitively, assets with large Sharpe ratios imply a large amount of compensation per unit of risk. This is great for investors, but it is not clear why there needs to be so much risk compensation in bond markets, especially the amount implied by the return-forecasting factor in the linear factor model. Moreover, it is unclear what *type of risks* the bondholder is being exposed to and thus demands compensation for because factor models have no economic content.

Figure 2: Model-Implied Expected Exchange Rate Changes



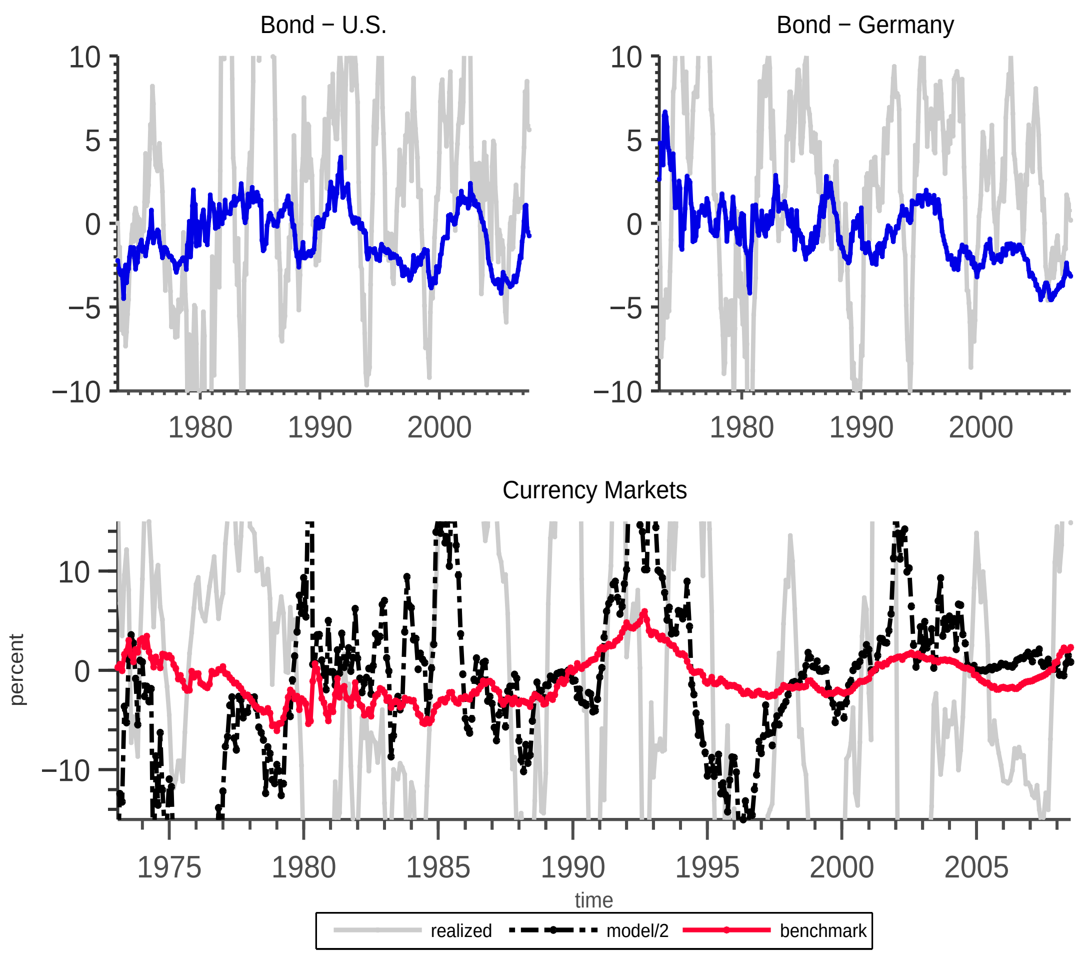
Note: The light grey line represents the realized 12-month exchange rate change in the dollar relative to the German deutsche mark (and after 1/1999 the Euro dollar). The monthly frequency data range from 1/1973-5/2009 and are expressed in percentages.

Figure 3: Model-Implied Currency Risk-Premia



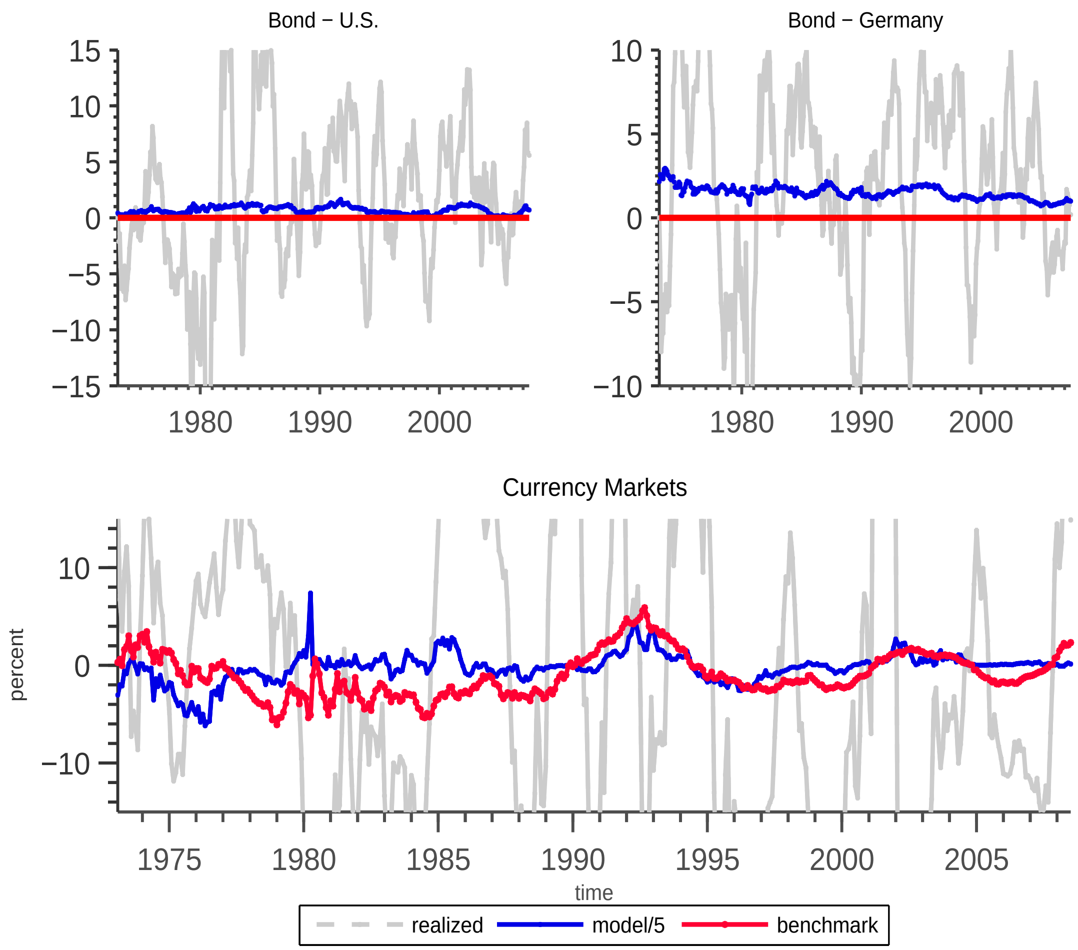
Note: The light grey line represents the realized 12-month currency excess return in the dollar relative to the German deutsche mark (and after 1/1999 the Euro dollar). The monthly frequency data range from 1/1973-5/2009 and are expressed in percentages.

Figure 4: Unaltered Model



Note: The light grey line represents the realized excess return for the respective bond market and currency market. The model generated *expected* excess returns are represented in (solid) blue and black (-.-.) line for the bond markets and currency markets, respectively. For currency markets, the (solid) red line represent the interest differential. The The monthly frequency data range from 1/1973-5/2009 and are expressed in percentages.

Figure 5: Modified Model



Note: The light grey line represents the realized excess return for the respective bond market and currency market. The model generated *expected* excess returns *after* reducing the bond risk prices by 5 are represented in (solid) blue line for both the bond markets and currency market. The solid (red) line represents the case where (a) there is no predictability in bond markets and (b) currency market prices are consistent with the forward premium anomaly in terms of the size of the predictable component found in the data that is *consistent* with our understanding of financial markets. The monthly frequency data range from 1/1973-5/2009 and are expressed in percentages.