Constrained Efficiency with Search and Information Frictions

S. Mohammad R. Davoodalhosseini*

November 20, 2014

Abstract

I characterize the constrained efficient (or planner's) allocation in a directed (competitive) search model with private information. There are sellers with private information on one side of the market and homogeneous buyers on the other side. They match bilaterally in different submarkets and trade. In each submarket, there are search frictions. In the market economy, homogeneous buyers enter different submarkets (i.e., post different contracts) and sellers with private information direct their search toward their preferred submarket.

I define a planner whose objective is to maximize social welfare subject to the information and matching frictions of the environment. The planner can impose taxes and subsidies on agents that vary across submarkets while being subject to an overall budget-balance condition. I show that the planner generally achieves strictly higher welfare than the market economy. I also derive conditions under which the planner achieves the complete information allocation. I present examples in the context of financial and labor markets, explicitly solve for the efficient tax and transfer schemes and compare the planner's allocation with the equilibrium allocation.

Keywords: Directed search, constrained efficiency, private information, free entry, cross-subsidization.

JEL: D82, D83, E24, G1, J31, J64

*Penn State University, sxd332@psu.edu. I would like to thank Neil Wallace for his guidance, encouragement, and tremendous support. I am also indebted to Manolis Galenianos and Shouyong Shi for their substantial advice and guidance. I would like also to thank Venky Venkateswaran, Kalyan Chaterjee, Vijay Krishna, Ed Green, David Jinkins and the participants of the seminars at Penn State University and Cornell-Penn State macroeconomics workshop for their helpful comments and suggestions. All remaining errors are my own.

1 Introduction

There are search frictions and private information in asset, labor, housing and other markets. For example consider markets for assets which are traded over the counter (OTC). It is natural to think that sellers have private information about the value of their assets. Also, they must incur search costs to find buyers for their assets. One important question that naturally arises is that whether the equilibrium in such economies is constrained efficient or not. More generally, what can a planner with the same information and search frictions present in the market economy achieve?

This paper studies the constrained efficient allocation in economies with directed (competitive) search and private information. There is a large number of homogeneous buyers on one side of the market whose population is endogenously determined through free entry. There is a fixed population of sellers on the other side of the market who have private information about their types. Buyers and sellers match bilaterally and trade in different locations, called submarkets. In each submarket, there are search frictions, in the sense that buyers and sellers on both sides get matched generally with probability less than one. Guerrieri, Shimer and Wright (2010) characterize equilibrium in this environment. I define and characterize the constrained efficient allocation in this environment and show that the equilibrium is generally constrained inefficient when the equilibrium fails to achieve the complete information allocation. I also derive sufficient conditions under which the planner can achieve the complete information allocation[1].

To define constrained efficient allocation, I consider a planner whose objective is to maximize social welfare. The planner chooses a set of submarkets for agents as locations for trade. He might choose some submarkets which would be inactive in the market economy, or he might shut down some submarkets which would be active in the market economy. He can also impose taxes and subsidies on agents. The planner, however, is subject to a budget balance condition. That is, the net amount of transfers that the planner makes to agents over all submarkets must be non-positive. The equilibrium allocation is a feasible allocation for the planner, because the revenues that the planner makes over **each** submarket is zero in the equilibrium allocation.

The planner faces the same information and search frictions present in the market economy. The planner cannot observe types of sellers and also cannot force sellers to participate

[1] In three examples, Guerrieri et al. (2010) introduce some pooling or semi-pooling allocations that Pareto dominate the equilibrium allocations. They do not characterize the constrained efficient allocation nor do they define that.

in the planner's allocation. Also, the net revenue of buyers from entry to each submarket which is chosen by the planner must be exactly zero. (This condition is equivalent to the free entry condition in the market economy.) In the language of mechanism design, the planner faces incentive compatibility of sellers and participation constraints of sellers and buyers and his own budget balance condition.

To understand how the planner can achieve strictly higher welfare than the market economy, let's consider the first example that I study in Section 4. Sellers have one indivisible asset which is of two types: high and low. The high type asset is more valuable both to buyers and sellers. Guerrieri et al. (2010) show that there exists a unique separating equilibrium in which different types trade in different submarkets. High type sellers prefer the higher price submarket with lower probability of matching (submarket two), because the asset is worth more to them in case of being unmatched. Low type sellers are just indifferent between the two submarkets. In fact, probability of matching here is used as a screening device.

The planner can do better than equilibrium in the following way: Beginning from the equilibrium allocation (which is feasible for the planner), the planner subsidizes sellers in submarket one (low type sellers) just a little bit so that their incentive compatibility constraint for choosing submarket two becomes slack. Now more buyers enter submarket two to get matched with already unmatched high type sellers. Therefore the welfare increases due to the formation of new matches. To finance subsidies to sellers in submarket one (low type sellers), the planner taxes sellers in submarket two (high type sellers). The planner keeps doing that until he achieves the complete information allocation, in which high type sellers also get matched with probability one, or participation constraint of high type sellers binds. The same idea goes through even if there are more than two types.

To understand the nature of inefficiency in the market economy, consider the externalities implied by having one more buyer in a submarket. First, it decreases the probability that other buyers are matched in that submarket. Second, it increases the probability that other sellers are matched in that submarket. In the presence of complete information, it is a well established result in the literature that buyers entering the market in directed search settings can internalize these externalities by choosing the "right" price (contract), if sellers' types are observable and contractible and if buyers can commit to their postings[2]. However, the

[2] This is probably the most important result in this literature, because this is in stark contrast with the result in random search settings in which the entrants generally fail to internalize these externalities. See the following papers for the efficiency properties of directed search: Moen (1997), Acemoglu and Shimer (1999), Shi (2001), Mortensen and Wright (2002), Shi (2002), Shimer (2005) and Eeckhout and Kircher (2010). See

change in the payoff of sellers following the entry of one more buyer in a submarket has another effect in this environment which is absent in the complete information case. This change will affect the incentive compatibility constraints that buyers who want to attract other types of sellers face, thus affecting the set of feasible submarkets that other buyers can enter to attract those sellers. Buyers in the market economy do not take into account the effect of their entry on the contracts posted on other submarkets and consequently on the payoff of sellers in other submarkets. The planner internalizes these externalities and therefore can do better than the market economy. The extent to which the planner can improve efficiency depends on the details of the environment. In my second result, I derive sufficient conditions under which the planner can eliminate distortions completely and achieve the complete information allocation.

In the second example in Section 5, I characterize the constrained efficient allocation in a version of the rat race (Akerlof (1976)) and compare my results with Guerrieri et al. (2010) who solve for the equilibrium allocation in this environment. There are two types of workers. Type two workers incur less cost for working longer hours and generate higher output compared to type one workers. Also the marginal output with respect to hours of work that type two workers generate is higher. In equilibrium, type two workers works inefficiently for longer hours than they would work under complete information and get matched with inefficiently higher probability. The planner, in contrast to the market economy, achieves the complete information allocation. He pays type one workers higher wages and type two workers lower wages than what they would get under complete information. These subsidies (to type one) and taxes (on type two) are needed to ensure that type one workers do not have any incentive to apply to the submarket that type two workers apply to. Moreover, if the share of type two workers in the population is sufficiently small, then the planner's allocation even Pareto dominates the equilibrium allocation.

Recently, Chang (2012) and Guerrieri and Shimer (2014) have used a directed search model with adverse selection to explain the sharp declines in asset prices and liquidity of assets in the recent financial crises. I study in Section 6 an asset market which is a static version of Chang (2012). I derive conditions under which the planner can achieve the complete information allocation by taxing the sellers in high price submarkets and subsidizing sellers in low price submarkets (similar to the simple two-type example). I characterize the parameter region in which the planner's allocation is separating. I show that in general this region is different than that in which the equilibrium allocation is separating. In particu-

Mortensen and Pissarides (1994) for a random search model.

lar, I show that the planner might want to separate types in the region of parameters that fire-sales occur in Chang (2012). Fire-sales in her paper is characterized by an equilibrium in which low type sellers and also those high type sellers who need liquidity sell their assets quickly in one submarket with a low price.

Guerrieri (2008) and Moen and Rosén (2011) study constrained efficient allocation in environments with directed search and private information. Specifically, Guerrieri (2008) shows that the competitive search equilibrium is constrained inefficient in a dynamic setting, if the economy is not on the steady state path. However in both papers, the agents who search (workers) do not have ex-ante private information. After they match with firms, they learn their types which become their private information.

Delacroix and Shi (2013) study a model in which sellers with private information post contracts (in contrast to Guerrieri et al. (2010) in which the uninformed side of the market posts contracts). They investigate the potentially conflicting roles of prices: the signaling role and the search directing role. Aside from some details[3], the notion of constrained efficiency defined in this paper and the ideas behind that (that the planner chooses some submarkets and makes appropriate transfers to agents) apply to their model as well, because the environments are similar, although they have a different trading mechanism.

The paper is organized as follows. In Section 2, I develop the environment of the model and define the planner's problem. In Section 3, I characterize the planner's allocation and state my main results. In Section 4, I study a two-type asset market example, characterize the planner's allocation and compare it with the equilibrium allocation. I explain the nature of inefficiency in the market economy and discuss why and how the planner can allocate resources more efficiently than the market economy. In Section 5, I study a version of the rat race. In Section 6, I study an asset market with a continuous type space and compare my results with Chang (2012). Section 7 concludes. All proofs appear in the appendix.

2 The Model

2.1 Environment

Consider an economy with two types of agents, buyers and sellers and $n+1$ goods where $n \in \mathbb{N}$. Goods $1, 2, ..., n$ are produced by sellers and consumed by buyers, while good $n+1$ is a numeraire good and produced and consumed by everyone. Let $a \equiv (a^1, a^2, ..., a^n) \in \mathbb{A} \subset \mathbb{R}^n$

[3]For example in their model, sellers choose the quality of their products. The quality then becomes their private information.

be a vector where $\mathbb{A}$ is compact, convex and non-empty. Component k of this vector, a^k, denotes the quantity of good k. For example in a labor market, a can be a positive real number denoting the hours of work. When I say an agent produces (or consumes) a, I mean that the agent produces (or consumes) a^1 units of good 1, a^2 units of good 2 and so on.

There is a measure 1 of sellers. A fraction $\pi_i > 0$ of sellers are of type $i \in \{1, 2, ..., I\}$. Type is seller's private information. On the other side of the market, there is a large continuum of homogenous buyers who can enter the market by incurring cost $k > 0$. After buyers enter the market, buyers and sellers are allocated to different submarkets (described below). Matching is bilateral. After they match, they trade.

There are search frictions in this environment. By search frictions I mean that sellers generally get to match with the buyers they have chosen with probability less than one. Matching occurs in submarkets which are simply some locations for trades. Matching technology determines the probability that sellers and buyers in each submarket get matched. If the ratio of buyers to sellers in one submarket is $\theta \in [0, \infty]$, then the buyers are matched with probability $q(\theta)$. Symmetrically, matching probability for sellers is $m(\theta) \equiv \theta q(\theta)$. As is standard in the literature, I assume that m is non-decreasing and q is and non-increasing. Both $m(.)$ and $q(.)$ are continuous.

Sellers' and buyers' payoff functions are quasi-linear in the numeraire good[4]. The payoff of a buyer, who enters the market, from consuming a and producing $t \in \mathbb{R}$ units of the numeraire good is $v_i(a) - t - k$ if matched with a type i seller and is $-k$ if unmatched. The payoff of a type i seller from producing a and consuming $t \in \mathbb{R}$ units of the numeraire good is $u_i(a) + t$ if matched with a buyer and is 0 otherwise.

2.2 The Planner's Problem

Before I describe the planner's problem, let me briefly describe how the market economy works, the especial case in which the planner does nothing. Submarkets in the market economy are characterized by (a, p) where $a \in \mathbb{A}$ denotes the vector of goods 1 to n to be produced by sellers in this submarket and $p \in \mathbb{R}$ is the amount of the numeraire good to be transferred from buyers to sellers. No submarket which can deliver buyers a strictly positive

[4]The difference between payoff functions here and in Guerrieri et al. (2010) is that I assume quasi-linear preferences for both sellers and buyers, while they do not make such an assumption. The reason that I impose quasi-linearity assumption is that I want to do welfare analysis and I want to use taxes and subsidies. If the preferences are not quasi-linear, the weight that the planner assigns to buyers and sellers might become important.

payoff is inactive in the equilibrium. If there was such a submarket, some buyers would have already entered that submarket to exploit that opportunity. On the other side of the market, sellers observe all (a, p) pairs posted in the market, anticipate the market tightness at each submarket, and then direct their search toward one which delivers them the highest expected payoff.

I define a planner whose objective is to maximize the weighted average of payoff to sellers[5]. This planner faces the same information and search frictions present in the market economy. To elaborate, the planner chooses a set of submarkets which are characterized by 3 elements, $(a, p, t_m) \in Y \equiv \mathbb{A} \times \mathbb{R}^2$, where a denotes the vector of goods to be produced by sellers, p denotes the amount of the numeraire good to be transferred from buyers to sellers, and t_m denotes the amount of the numeraire good to be transferred from buyers to the planner conditional on trade. Also, the planner pays $T \in \mathbb{R}_+$ units of the numeraire good to the sellers who do not apply to any submarket[6]. Note that any post in the market economy is a special case of this description with $t_m = 0$ for every submarket and $T = 0$.

The planner, in contrast to the market economy, is assumed to have the power to pick any selection of submarkets for agents as possible locations for trade. He might select some submarkets which would be inactive in the market economy, or he might shut down some submarkets which would be active in the market economy. He is also assumed to have the power to impose taxes and subsidies on agents. Aside from these two powers (picking submarkets and possibility of transfers), the market economy and the planner face the same restrictions: None can condition amounts of goods to be produced or payments of the numeraire good on the types of sellers. Ex-ante payoffs of buyers in both cases should be 0 to ensure that buyers want to participate and also to ensure that there is no excess entry into any submarket. Also in both cases sellers choose submarkets which maximize their expected payoff or stay out. (If they get a negative expected payoff, they will not participate.)

The planner faces a budget constraint, or a budget balance condition as called in the mechanism design literature. This condition states that the net amount of transfers that the planner makes to agents should not exceed 0. Notice that in the market economy, it is not possible to transfer funds (the numeraire good) from one submarket to another. That is, all

[5]Note that buyers get 0 payoff either in the market economy or under the planner's allocation

[6]It is possible that some types might get a negative payoff from active sub-markets, so they prefer not to apply to any submarket. However they distort the allocation for other types via incentive compatibility. The planner is allowed to pay them T to reduce the distortion. I assume that T is positive, because the planner cannot force sellers to get a negative payoff.

the surplus generated in any submarket belongs to sellers in that submarket. The planner, on the other hand, might allocate sellers more or less payoffs than the surplus they generate.

2.3 Formal Definition of the Planner's Problem

First I define an allocation and then a feasible allocation. Then I define the constrained efficient allocation or the planner's allocation. Let $y \equiv (a, p, t_m)$ denote a submarket. To make the notation clear, the first component of y is denoted by a, rather than y_1. The second and third components are respectively denoted by p and t_m. Similarly if the submarket is denoted by y', then the first, second and third components of y' are denoted by a', p' and t'_m. Let $\gamma_i(y)$ denote the share of sellers that are type i in the submarket denoted by y, with $\Gamma(y) \equiv \{\gamma_1(y), ..., \gamma_i(y), ..., \gamma_I(y)\} \in \Delta^I$. Δ^I is an $I-$dimensional simplex, that is, for all y, $0 \leq \gamma_i(y) \leq 1$ and $\sum_{i=1}^{I} \gamma_i(y) = 1$.

An allocation $\{Y^P, \lambda, \theta, \Gamma, T\}$ is a set of open submarkets $Y^P \subseteq Y$, the distribution of buyers over open submarkets λ, the ratio of buyers to sellers for each open submarket $\theta : Y^P \rightarrow [0, \infty]$, the distribution of types in each open submarket Γ and finally the amount of the numeraire good transferred to sellers who do not apply to any submarket $T \in \mathbb{R}_+$.

Because the planner faces some constraints, only some allocations are feasible for the planner:

Definition 1. *A planner's allocation $\{Y^P, \lambda, \theta, \Gamma, T\}$ is feasible if it satisfies the following conditions:*

1. *(Sellers' maximization) Let $w_i \equiv \max_{y' \in Y^P}\{m(\theta(y'))(u_i(a') + p')\}$. Given i,*

   (a) *if $w_i < T$, then $\gamma_i(y) = 0$ for all $y \in Y^P$,*

   (b) *if $w_i \geq T$, for any $y \in Y^P$ and i such that $\gamma_i(y) > 0$,*

   $$y \in \arg\max_{y' \in Y^P} \left\{ m(\theta(y'))(u_i(a') + p') \right\},$$

   (c) *and $\int_{Y^P} \frac{\gamma_i(y)}{\theta(y)} d\lambda(\{y\}) \leq \pi_i$, with equality if $w_i > T$.*

2. *(Buyers' zero profit) For any $y \in Y^P$,*

   $$q(\theta(y)) \sum_i \gamma_i(y)(v_i(a) - p - t_m) = k.$$

3. (Planner's budget constraint)

$$\int_{Y^P} q(\theta(y)) t_m d\lambda(\{y\}) \geq (1 - \int_{Y^P} \frac{1}{\theta(y)} d\lambda(\{y\}))T.$$

Sellers' maximization summarizes two constraints: sellers' participation (individual rationality) constraint and sellers' incentive compatibility constraint. The former implies that if type i sellers get a payoff less than T from the open submarkets, they will choose not to apply to any submarket and get a payoff of T. If they get a payoff more than T, they apply to some submarkets. If their payoff is exactly equal to T, then they are indifferent between participation in the allocation and their outside option.

For any allocation, let X_i be defined as follows: $X_i \equiv \{(\theta(y), a) \big| y \equiv (a, p, t_m) \in Y^P, \gamma_i(y) > 0\}$. Denote elements of X_i by (θ_i, a_i). In words, θ_i is the market tightness of a submarket to which type i applies with positive probability and a_i is the production level at that submarket. Incentive compatibility (IC) constraint implies that for any i, j, $(\theta_i, a_i) \in X_i$ and $(\theta_j, a_j) \in X_j$:

$$m(\theta_i)(u_i(a_i) + p_i) \geq m(\theta_j)(u_i(a_j) + p_j) \text{ (IC)}.$$

With respect to the second condition, if buyers' expected payoff in one submarket is strictly negative, no buyer enters that submarket. If it is strictly positive, more buyers will enter that submarket. Therefore, for all markets that the planner wants to be open, buyers must get exactly 0 expected payoff. A buyer has to incur entry cost k if he wants to enter submarket y. Then, he gets matched with a type i seller with probability $\gamma_i(y)$ from which he gets a payoff of $v_i(a)$ in terms of the numeraire good, and pays p units of the numeraire good to the seller and t_m to the planner. The last constraint states that the net amount of the numeraire good that the planner collects from buyers in different submarkets must be enough to cover the subsidies to sellers who do not apply to any submarket. Among all feasible allocations, the planner chooses one that maximizes his objective functions, which is just sum of payoff to sellers. Let U_i denote the payoff of type i from allocation $\{Y^P, \lambda, \theta, \Gamma, T\}$. Then, U_i is claculated as follows:

$$U_i \equiv \max\{T, \max_{y' \in Y^P} \{m(\theta(y'))(u_i(a') + p')\}\}.$$

Definition 2. *A Constrained efficient allocation is a feasible allocation* $\{Y^P, \lambda, \theta, \Gamma, T\}$ *which maximizes welfare among all feasible allocations:*

$$\max \sum_i \pi_i U_i$$

Subject to: $\{Y^P, \lambda, \theta, \Gamma, T\}$ *is feasible.*

I could assume alternatively that the planner uses a direct mechanism to allocate resources. In the direct mechanism, and thanks to the revelation principle, sellers report their types to the planner and the planner assigns type i sellers a 4-tuple $(a_i, p_i, s_i, \theta_i)$, where the first element is the level of production by type i, the second element is the amount of the numerarire good given to type i if matched with some buyers, the third element is the amount of the numerarire good given to type i if unmatched, and θ_i is the average number of buyers allocated to a type i seller (so the seller is matched with $m(\theta_i)$). The planner maximizes his objective function subject to incentive compatibility of sellers, individual rationality of sellers and buyers and his budget balance condition. It can be shown that the way I formulate the problem is without loss of generality. That is, a planner who uses a direct mechanism achieves the same welfare as the the welfare that the planner in my formulation achieves. The reason that I did not choose that formulation is that implementing a direct mechanism in the real world requires a large amount of communication. Rather, the way I formulate the problem is closer to the real world applications. The planner's allocation in my model can be implemented in the market by only imposing appropriate scheme of taxes and subsidies on agents.

2.4 Equilibrium Definition

I define the equilibrium here in order to be able to compare the equilibrium allocation with the planner's allocation. The definition of equilibrium is taken from Guerrieri et al. (2010). Let $\bar{Y}$ be defined as $\bar{Y} \equiv \cup_i \bar{Y}_i$, where

$$\bar{Y}_i \equiv \big\{(a, p, 0)\big|(a, p) \in \mathbb{A} \times R,\ q(0)(v_i(a) - p) \geq k, \text{ and } u_i(a) + p \geq 0\big\}.$$

If $(a, p, 0) \notin \bar{Y}$, then no type will be attracted to this submarket, therefore we restrict our attention to only $(a, p, 0)$ in $\bar{Y}$. Note that for any $y \in \bar{Y}$, the third element is 0, because there are no transfers between the planner and agents in the equilibrium.

Definition 3 (Guerrieri et al. (2010))**.** *An equilibrium $\{Y^{eq}, \lambda^{eq}, \theta^{eq}, \Gamma^{eq}\}$, is a measure λ^{eq} on $\bar{Y}$ with support Y^{eq}, a function $\theta^{eq} : \bar{Y} \to [0, \infty]$, and a function $\Gamma^{eq} : \bar{Y} \to \Delta^I$ which satisfies the following conditions:*

1. *(Sellers' optimal search) Let $U_i^{eq} = \max\Big\{0, \max_{y' \in Y^{eq}} \big\{m(\theta^{eq}(y'))(u_i(a') + p')\big\}\Big\}$ and $U_i^{eq} = 0$ if $Y^P = \emptyset$. Then for any $y \in Y$ and i, $U_i^{eq} \geq m(\theta^{eq}(y'))(u_i(a') + p')$ with equality if $\theta^{eq}(y) < \infty$ and $\gamma_i^{eq}(y) > 0$. Moreover, if $u_i(a) + p < 0$, either $\theta^{eq}(y) = \infty$ or $\gamma_i^{eq}(y) > 0$.*

2. *(Buyers' profit maximization and free entry) For any* $y \in \bar{Y}$,

$$q(\theta^{eq}(y)) \sum_i \gamma_i^{eq}(y)(v_i(a) - p) \leq k,$$

with equality if $y \in Y^{eq}$.

3. *(Market clearing) For all* i, $\int_{Y^{eq}} \frac{\gamma_i^{eq}(y)}{\theta^{eq}(y)} d\lambda^{eq}(\{y\}) \leq \pi_i$, *with equality if* $U_i^{eq} > 0$.

Given any equilibrium $\{Y^{eq}, \lambda^{eq}, \theta^{eq}, \Gamma^{eq}\}$, I construct an allocation called **equilibrium allocation** $\{Y^{eq}, \lambda^{eq}, \theta, \Gamma, T\}$ where $\theta : Y^{eq} \to [0, \infty]$, $\Gamma : Y^{eq} \to \Delta^I$ and $\theta(y) = \theta^{eq}(y)$, $\Gamma(y) = \Gamma^{eq}(y)$, for all $y \in Y^{eq}$ and $T = 0$. One difference between equilibrium and equilibrium allocation is that in equilibrium, the market tightness and the composition of types must be defined over all submarkets (for all $y \in \bar{Y}$), because when buyers want to enter the market, they need to form beliefs about all submarkets, either in the support of λ^{eq} or not. In contrast for the equilibrium allocation, we just need to define them over active submarkets ($y \in Y^{eq}$), because buyers cannot enter any other submarket. Also, there is no transfer to buyers at any submarket, $t_m = 0$ for all submarkets, and also no transfer to the sellers who don't apply to any submarket, $T = 0$.

The equilibrium allocation is feasible, because sellers' maximization condition and buyers' zero profit condition are satisfied following their counterparts in the definition of equilibrium. The planner's budget constraint is also trivially satisfied, because $t_m = 0$ for all $y \in Y^{eq}$. When I say equilibrium allocation, I mean the feasible allocation which is constructed from equilibrium objects as above. Finally, when I refer to equilibrium in the paper, I mean the notion of equilibrium where uninformed side (buyers) posts contracts. I do not mean a notion of equilibrium with signaling in which the informed side of the market posts contracts, unless otherwise noted.

3 Characterization

As a benchmark, I first study complete information case and then I present my main results.

3.1 Complete Information Allocation

As a benchmark and for the future reference, consider the market economy when the type of sellers is common knowledge. The submarkets, therefore, are also indexed by the type of sellers that the buyers want to meet. The buyers, who contemplate what submarket to enter to attract type i sellers, enter a submarket which maximizes the payoff of type i subject to the free entry condition. (See Moen (1997) for further explanation.) If there is any submarket that would deliver type i sellers a higher payoff, some buyers would enter that submarket and then, sellers would strictly prefer that submarket. Therefore buyers who attract type i solve the following problem in the market economy with complete information:

$$\max_{\theta,a,p}\{m(\theta)(p+u_i(a))\}$$
$$\text{s.t. } q(\theta)(v_i(a)-p)=k.$$

Denote the solution to this problem by $(\theta_i^{CI}, a_i^{CI}, p_i^{CI})$. Eliminating p from the maximization problem, one can write the payoff of type i sellers from participating in the market in the complete information case as follows:

$$U_i^{CI}=\max_{\theta,a}\{m(\theta)(v_i(a)+u_i(a))-k\theta\}.$$

Notice that the function in the maximization problem, $m(\theta)(v_i(a)+u_i(a))-k\theta$, is exactly equal to the surplus created by type i sellers. Thus, the planner who observes types of sellers solves exactly the same problem as buyers in the market economy. (If $U_i^{CI}<0$, then type i sellers do not participate in the market. The planner does not want them to participate, either.) If $U_i^{CI}\geq 0$, the planner wants type i to get matched with probability $m(\theta_i^{CI})$ and to produce a_i^{CI}. This is the core of the argument in the literature which states that the market economy decentralizes the planner's allocation under complete information[7]. In this paper, when I say that the planner *achieves the complete information allocation*, I mean that there exists a feasible allocation in which type i sellers get matched with probability $m(\theta_i^{CI})$ and produce a_i^{CI}.

3.2 Results

We have already seen that the equilibrium allocation is feasible for the planner. It is immediately followed that the planner can achieve at least the level of welfare in the market economy. The following proposition states that the planner can achieve strictly higher welfare.

Assumption 1. *1. Strict Monotonicity: For all $(a,.,.)\in\bar{Y}$, $v_1(a)<v_2(a)<...<v_I(a)$.*

2. Sorting: For all i, $(a,.,.)\in\bar{Y}$ and $\epsilon>0$, there exists an $a'\in B_\epsilon(a)$ $\equiv\{a'\in\mathbb{A}|\,||a-a'||_2<\epsilon\}$ such that

$$u_j(a')-u_j(a)<u_h(a')-u_h(a) \text{ for all } j \text{ and } h \text{ with } j<i\leq h.$$

[7] As already stated, there are many papers with this message in the literature, such as Moen (1997), Acemoglu and Shimer (1999), Shi (2001), Shi (2002) and Shimer (2005).

Proposition 1 (Result 1)**.** *Suppose Assumption 1 holds. Also assume that all types with positive gains from trade (all i with $U_i^{CI} > 0$) are active in the equilibrium. If the market economy fails to achieve the complete information allocation, then the planner achieves* ***strictly*** *higher welfare than the market economy.*

A standard single crossing condition states that the indifference curves of different types must cross only once. The sorting assumption here (which is the same as in Guerrieri et al. (2010)) is in a sense a local crossing condition, because it allows that a' is greater than a for some a and less than a for other a. Moreover, it is in a sense stronger than single crossing condition, because it states given any a, there *exists* an a' with such a property. The requirement that all types with positive gains from trade must be active in the equilibrium is satisfied if there are positive gains from trade for all types. In an example in Section 4, I will make it clear why this assumption is necessary for the result.

The idea of the proof is as follows. Since the equilibrium allocation is feasible for the planner, I begin from that allocation and propose a set of transfers to improve upon that allocation. We need first to understand how the equilibrium is constructed. Under similar conditions (weak monotonicity and sorting), Guerrieri et al. (2010) prove that the equilibrium for type i is characterized by maximizing the payoff of type i, subject to the free entry condition and incentive compatibility constraints of all lower types. That is, type $j < i$ does not get a higher payoff if he chooses the submarket that type i chooses. They prove that this equilibrium is unique in terms of payoffs. Let $\{Y^{eq}, \lambda^{eq}, \theta^{eq}, \Gamma^{eq}, T^{eq}\}$ denote the equilibrium allocation where $Y^{eq} \equiv \{y_1^{eq}, y_2^{eq}, ..., y_I^{eq}\}$ and U_i^{eq} denote the utility that type i gets in the equilibrium.

In this explanation, assume that all types are active in the equilibrium, $U_i^{eq} > 0$. If any IC constraint is binding in the equilibrium, say type j is indifferent between y_j^{eq} and y_i^{eq} with $j < i$, then the planner subsidizes all types lower than i by some small amount $\epsilon > 0$ in a lump sum way: He increases p at each submarket y_h $(h < i)$ by $\frac{\epsilon}{m(\theta(y_h))}$ amount. Now constraints of the maximization problem for type i become slack, so the planner can find (θ', a') which increases the welfare generated by type i. Therefore, the payoff of type i strictly increases. Now consider type $i+1$. The planner solves the maximization problem for type $i+1$ again. Since all lower types including type i get a strictly higher payoff than the equilibrium allocation now, the maximization problem for type $i+1$ is now less constrained, so the planner can achieve weakly higher welfare for type $i+1$ as well. The planner keeps doing the same thing for all types above i and assigns them new (θ, a) pairs. So far the welfare of the population has increased. Then the planner, given (θ, a) of every submarket, adjusts payments to buyers as well to make buyers' zero profit condition satisfied. Finally, the planner distributes the transfers (in a positive or negative amount) back to all agents in a lump-sum way so that IC is not distorted. Adjusting payments to buyers and making transfers to all agents does not change the welfare of the population, therefore, the welfare now is strictly higher than the level of welfare in the equilibrium (because type i has generated strictly higher surplus and all types $i+1$ to I have generated weakly higher surplus).

In the next proposition, I provide sufficient conditions for the planner to achieve the complete information allocation.

Assumption 2. *Let $a \equiv (a^1, a^2, ..., a^n)$ and $i \in \{1, 2, ..., I\}$ and $k \in \{1, 2, ..., n\}$.*

1. *Monotonicity of u in i: $u_1(a) \leq u_2(a) \leq ... \leq u_I(a)$ for all $a \in \mathbb{A}$.*
2. *Single crossing of u in $(a; i)$: $u_i(a) - u_{i-1}(a)$ is increasing in a^k for all $a \in \mathbb{A}$, $i \geq 2$ and k.*
3. *Single crossing of $u+v$ in $(a; i)$: $u_i(a) + v_i(a) - (u_{i-1}(a) + v_{i-1}(a))$ is increasing in a^k for all $a \in \mathbb{A}$, $i \geq 2$ and k.*
4. *Supermodularity of $f_i \equiv u_i + v_i$ in a:*

$$f_i(a''') + f_i(a) \geq f_i(a') + f_i(a'') \text{ for all } i \text{ and } a, a', a'', a''' \in \mathbb{A},$$

   where $a \equiv (a^1, a^2, ..., a^n)$, $a' \equiv (a^1, ..., a^{k-1}, b^k, a^{k+1}, ..., a^n)$, $a'' \equiv (a^1, ..., a^{l-1}, b^l, a^{l+1}, ..., a^n)$, and $a''' \equiv (a^1, ..., a^{k-1}, b^k, a^{k+1}, ..., a^{l-1}, b^l, a^{l+1}, ..., a^n)$, and $b^k \geq a^k$, $b^l \geq a^l$ and $k < l$.

5. *Either (a) holds or (b) and (c) hold:*
   (a) *Monotonicity of v in i: $v_1(a) \leq v_2(a) \leq ... \leq v_I(a)$ for all $a \in \mathbb{A}$.*
   (b) *Monotonicity of f in i: $f_1(a) \leq f_2(a) \leq ... \leq f_I(a)$ for all $a \in \mathbb{A}$.*
   (c) *Sufficient gains from trade for all types:*

$$U_i^{CI} \geq m(\theta_{i-1}^{CI})(u_i(a_{i-1}^{CI}) - u_{i-1}(a_{i-1}^{CI}))\frac{\sum_{j=i}^{I} \pi_j}{\pi_i}, \text{ for } i > 1 \text{ and } U_1^{CI} \geq 0.$$

Proposition 2 (Result 2)**.** *Under Assumption 2, the planner achieves the complete information allocation.*

The first part of the assumption simply states that the payoff of higher types is higher than lower types for any given level of production. The second part is a standard single crossing condition[8] which implies that given a level of production, the marginal *payoff* of higher types with respect to the level of production of good $k \in \{1, 2, ..., n\}$ is higher than that of lower types. Similarly, the third part states that the marginal *surplus* with respect to the level of production of good k that higher types create is higher than the associated marginal surplus that lower types create. The fourth part is a standard supermodularity condition which states that the marginal surplus created by type i with respect to the level of production of good k is increasing in the level of production of good l ($k \neq l$). In the last part, I require either of the two following conditions. For any given level of production, buyers weakly prefer higher types of sellers. If this assumption is not satisfied, I require that $u_i + v_i$ is increasing in i for any production level (in part 5(b)) and also $\frac{\sum_{j=i}^{I} \pi_j}{\pi_i}$ is less than some threshold for every $i > 1$.

The proof follows a guess and verify approach. Again for this explanation, assume that there are positive gains from trade for all types. I first guess that the planner can achieve the complete information allocation under the conditions in Assumption 2. Then I ensure that all conditions for feasibility are satisfied. The planner achieves the complete information allocation iff type i sellers get matched with probability $m(\theta_i^{CI})$ and produce a_i^{CI}. I need to find a set of transfers which together with the complete information allocation satisfies IC constraints. To find such transfers, I show that if part 1 of Assumption 2 holds and if transfers are such that local downward IC constraints are satisfied and are binding, then all IC constraints are satisfied. By local downward IC constraint I mean that type i does not gain by applying to the submarket that type $i-1$ applies to. Moreover, I show that the amount of transfers to the lowest type (p_1) determines the amount of transfers for all other types. There is a set of transfers that satisfies local downward IC constraints, if $(\theta_i^{CI}, a_i^{CI})$ is increasing in i and also if u satisfies single crossing condition (part 2 of Assumption 2)[9].

According to Milgrom and Shannon (1994), if $u_i + v_i$ satisfies parts 4 and 5 of Assumption 2, then $a_i^{CI} \equiv \arg\max_a \{u_i(a) + v_i(a)\}$ is increasing in i. If $u_i + v_i$ is increasing in i (part 3 of Assumption 2), then $\arg\max_\theta \{m(\theta)(u_i(a_i^{CI}) + v_i(a_i^{CI})) - k\theta\}$ will be also increasing in i, so $(\theta_i^{CI}, a_i^{CI})$ is increasing in i. Then the planner adjusts p_1 such that all types get a positive payoff. (This implies that $T = 0$). Given the transfer scheme to sellers, the planner adjusts

[8]This condition is also called Spence-Mirless single crossing property. See Milgrom and Shannon (1994) for a full discussion about this property.

[9]See Theorem 7.1 and 7.3 in Fudenberg and Tirole (1991) or Section 3.1 in Laffont and Martimort (2009) for reference.

payments to buyers such that the zero profit condition and the planner's budget constraint are satisfied. In future sections, I will make it clear by a couple of examples the mechanism through which the planner can improve the welfare upon the market economy and how he might achieve the complete information allocation.

4 Example 1: Asset Market with Lemons

So far I have considered a general framework. In the following two sections, I study two examples from Guerrieri et al. (2010) and characterize the constrained efficient allocation for them and compare them with the associated equilibrium allocations. At the end of this section, I provide some intuition about how and why the planner can increase welfare by using appropriate transfers. Also, I explain the nature of inefficiency in models of directed search with private information.

The first example is an asset market with lemons (in the spirit of Akerlof (1970)). There are two types of assets, with value c_i to the seller and h_i to the buyer. Both c_i and h_i are in terms of a numeraire good. The payoff of a buyer matched with a type i seller is $\alpha h_i - t - k$ where α is the probability that the buyer gets the asset from the seller and t is the amount of the numeraire good that he pays (either to the planner or to sellers) in terms of the numeraire good. The payoff of a type i seller matched with a buyer is $-\alpha c_i + t$ where α is the probability that the seller gives the asset to the buyer and t is the amount of the numeraire good he consumes. The buyer's payoff is $-k$ if unmatched. As a special case of the original setting, here: $I = 2$, $n = 1$, $a \equiv \alpha$, $u_i(\alpha) = -\alpha c_i$ and $v_i(\alpha) = \alpha h_i$. The matching function is $m(\theta) = \min\{1, \theta\}$, that is, the short side of the market gets matched for sure. Following Guerrieri et al. (2010), I also make the following assumptions:

Assumption 3. *In the asset market with lemons,*

1. $0 < h_1 < h_2$ *and* $0 < c_1 < c_2$.

2. *For* $i = 1, 2$, $c_i < b_i \equiv h_i - k$.

Proposition 3. *In the asset market with lemons, the planner achieves higher welfare than the market. If* $\pi_1 b_1 + \pi_2 b_2 \geq c_2$*, then the planner achieves the complete information allocation. See full characterization of the constraint efficient allocation in Table 1.*

This proposition at the first part is basically a special case of result 1 that the planner achieves higher welfare than the market economy. Then, in order to fully characterize the constraint efficient allcoation, I separate two cases: $\pi_1 b_1 + \pi_2 b_2 \geq c_2$ and $\pi_1 b_1 + \pi_2 b_2 < c_2$. Specially in the first case, I show that the planner achieves the complete information allocation. This claim is stronger than result 2. It is easy to check that in order for Assumption 2 to be satisfied[10], we need $b_1 \geq c_2$ which is a stronger requirement than the requirement in Proposition 3 that $\pi_1 b_1 + \pi_2 b_2 \geq c_2$, because $b_2 > b_1$ by assumption.

In the second and third columns of Table 1 the equilibrium outcomes under complete information and under private information are described respectively. In the fourth and fifth columns, I describe the planner's allocation under different conditions.

Since there are positive gains from trade for both types according to part 2 of Assumption 3, under complete information the planner wants both types to get matched with probability 1 ($\theta_1 = \theta_2 = 1$) and also trade with probability 1 ($\alpha_1 = \alpha_2 = 1$). As already discussed under complete information, the market decentralizes the efficient allocation.

In the equilibrium with private information, different types trade in different submarkets. In submarket one, price (p_1) is lower, but probability of matching is higher compared to submarket two. The market tightness is used as a screening device here. Putting it differently, the probability of matching for type two is distorted so that type 1 does not want to apply to submarket two, although the price is higher there. The equilibrium allocation is independent of the distribution of types.

If $\pi_1 b_1 + \pi_2 b_2 \geq c_2$, then the planner achieves the complete information allocation through a pooling allocation. (See the fourth column of Table 1). In this allocation, the planner does not need to use any transfers. All he needs to do is to restrict entry of buyers to other submarkets. This allocation cannot be sustained as equilibrium, because buyers would have incentives to open a new submarket with a higher price to attract only high type sellers from the pool. The planner does not open such a submarket, because in that case, the probability that high quality sellers get matched will be reduced compared to the complete information allocation.

Now assume that $\pi_1 b_1 + \pi_2 b_2 < c_2$. The planner's allocation in this case is reported in the fifth column of Table 1. Type 2 would get less than 0 under the pooling allocation, so pooling both types is not feasible. Therefore, the complete information allocation is not achievable via a pooling allocation. The complete information allocation is not achievable through any separating allocation as well, because if $\alpha_1 = \alpha_2 = \theta_1 = \theta_2 = 1$, then the payment to sellers in both submarkets should be the same to satisfy IC condition. If the

[10] Parts 1 to 4 and also part 5(b) are easily satisfied. I just need to check Assumption 2 part 5(c): $U_2^{CI} \geq 0$ and $U_1^{CI} \geq m(1)(-c_1 + c_2)$. The latter is equivalent to $b_1 > c_1$. Note that here, types one and two must be relabeled, so we can apply Proposition 2 to the asset market with lemons.

	Complete information	Equilibrium	Constrained efficient if $\pi_1 b_1 + \pi_2 b_2 \geq c_2$	Constrained efficient if $\pi_1 b_1 + \pi_2 b_2 < c_2$
α_1, α_2	1	1	1	1
θ_1	1	1	1	1
θ_2	1	$\frac{b_1-c_1}{b_2-c_1} < 1$	–	$\frac{\pi_1(b_1-c_1)}{c_2-\pi_1 c_1-\pi_2 b_2}$
p_1	b_1	b_1	$\pi_1 b_1 + \pi_2 b_2$	$\frac{\pi_1 b_1(c_2-c_1)+\pi_2 c_1(c_2-b_2)}{c_2-\pi_1 c_1-\pi_2 b_2}$
p_2	b_2	b_2	–	c_2
$t_{m,1}$	0	0	0	$-\frac{\pi_2(b_2-c_2)(b_1-c_1)}{c_2-\pi_1 c_1-\pi_2 b_2}$
$t_{m,2}$	0	0	–	$b_2 - c_2$
U_1	$b_1 - c_1$	$b_1 - c_1$	$\pi_1 b_1 + \pi_2 b_2 - c_1$	$\frac{\pi_1(b_1-c_1)(c_2-c_1)}{c_2-\pi_1 c_1-\pi_2 b_2}$
U_2	$b_2 - c_2$	$\frac{b_1-c_1}{b_2-c_1}(b_2 - c_2)$	$\pi_1 b_1 + \pi_2 b_2 - c_2$	0

Table 1: Comparison between different allocations in the asset market for lemons

payments in both submarkets are equal, then this allocation is pooling, but it is already shown that the pooling allocation is not feasible.

4.1 Explanation of the Results

To understand how the planner improves efficiency, assume as a thought experiment that the planner begins from the equilibrium allocation and wants to increase welfare. We have already seen that the equilibrium allocation is feasible for the planner. In equilibrium type one is indifferent between choosing submarket one and submarket two. Although there are some type two sellers unmatched in submarket two, buyers do not enter submarket two any more, because more entry will make submarket two strictly preferable for type one, thus leading to entry of type one to submarket two. Nevertheless, matching with type one sellers in submarket two with positive probability is not worthwhile for buyers given the high price that buyers need to pay in submarket two. In short, the IC constraints that buyers face do not allow more buyers to enter submarket two.

To improve efficiency, the planner increases the payment to type one (p_1) so that the IC of type one for choosing submarket two becomes slack (type one strictly prefers submarket one over submarket two). Now more buyers have incentives to enter submarket two to get matched with already unmatched sellers of type two . To finance subsidies to sellers

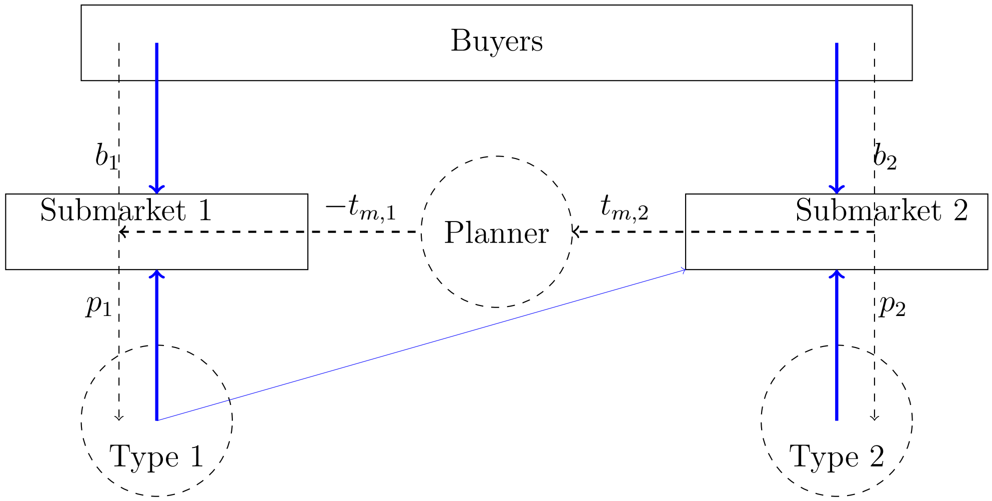

Figure 1: This schematic diagram illustrates how the planner allocates resources. Dashed lines show the flow of funds. In equilibrium, type one is indifferent between two submarkets. The planner taxes sellers in submarket two and subsidizes sellers in submarket one so that type one sellers are discouraged from applying to the higher price submarket. Now, more buyers enter submarket two and the outcome approaches the complete information allocation.

in submarket one (type one sellers), the planner must tax sellers in submarket two (type two sellers). The planner keeps increasing p_1 and decreasing p_2 until one of the following happens. Either he achieves the complete information allocation, which is the case in the pooling allocation where both types trade with probability 1, or participation constraints of type two sellers bind, that is, type two sellers get exactly 0 payoff. The former happens if $\pi_1 b_1 + \pi_2 b_2 \geq c_2$ and the latter happens if $\pi_1 b_1 + \pi_2 b_2 < c_2$. Figure 1 illustrates this point. Although I explained the main idea through a two-type example, the intuition is the same in the general n-type setting, or even in a continuous type one which will be discussed in Section 6.

The main difference between planer's allocation and the equilibrium allocation is that in the equilibrium, the payment to sellers is exactly equal to the payment that buyers make. Also, because of free entry condition, the buyers get 0, so the sellers get the whole surplus in every submarket. But it is feasible for the planner to give sellers in one submarket more and sellers in other submarkets less than the surplus they generate. The only constraint that the planner faces is the budget constraint over all submarkets. That is, the amount of transfers that buyers pay must be equal to amount of transfers that sellers receive over all submarkets.

Entry of more buyers in a submarket creates two types of externalities on others. First, it decreases the probability of matching of other buyers in that submarket. Second, it changes

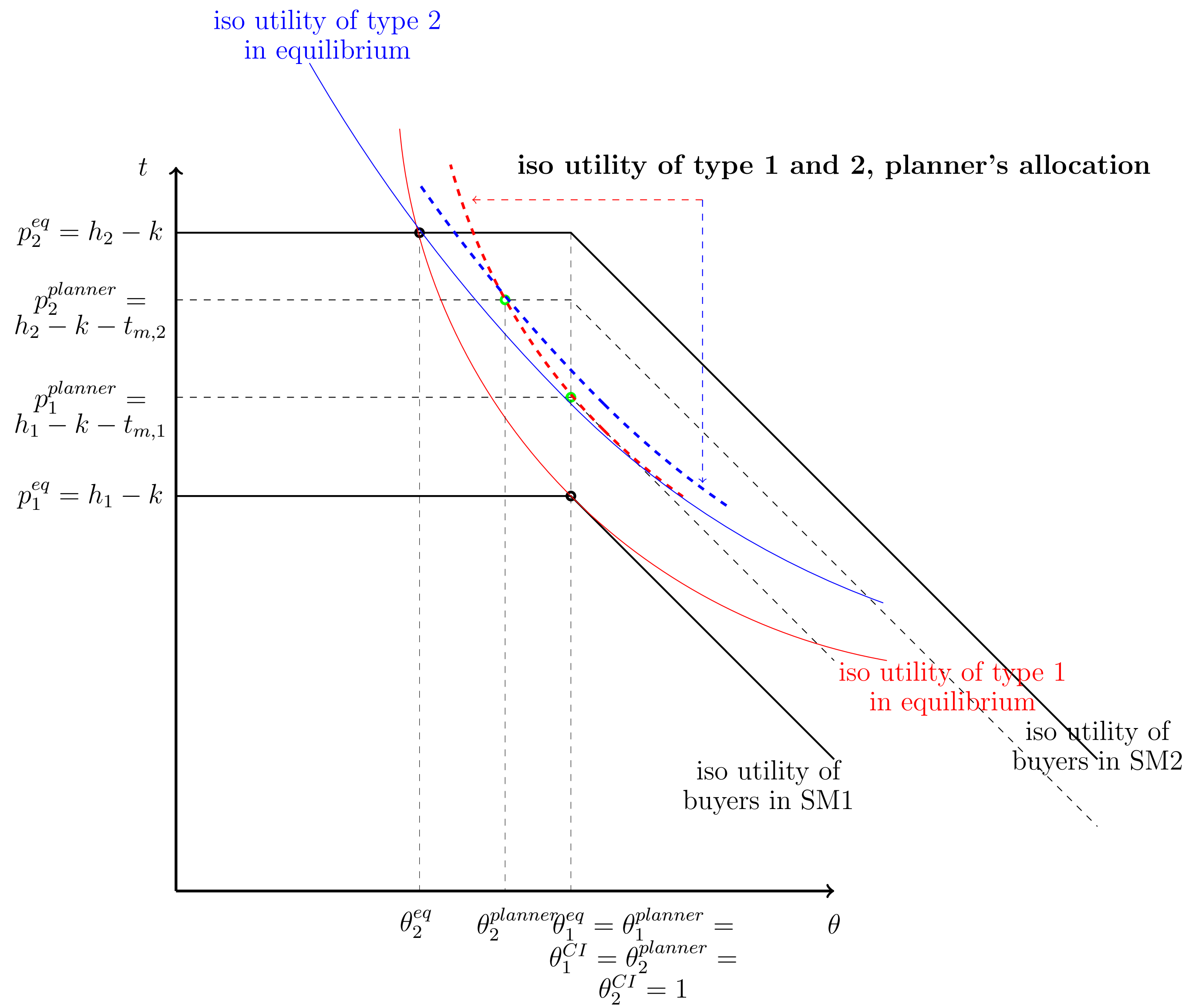

Figure 2: The indifference curves of buyers and sellers are illustrated here. In the equilibrium allocation, the market tightness for type two is less than 1. Intersection of indifference curve of type 1 and indifference curve of buyers in submarket two determines θ_2^{eq}. At this point, type one is indifferent between both submarkets. The planner makes subsidies to buyers at submarket one ($t_{m,1} < 0$), thus pushing buyers' indifference curves in that submarket upward. Because of zero profit condition for buyers, eventually type 1 sellers get a higher payoff than equilibrium. The planner taxes buyers in submarket two ($t_{m,2} > 0$) to raise funds for subsidies made to type one. Now, the market tightness that the planner assigns to type two is increased compared to that in equilibrium.

the payoff of sellers in that submarket. In the complete information case, the change in the payoff of sellers in one submarket does not affect the payoff of sellers in other submarkets. In fact, under complete information, the negative externality that entrants impose on other buyers is exactly offset by the amount of positive externalities that they impose on sellers and therefore the equilibrium allocation is efficient. Here, because there is private information, the change in the payoff of one type of sellers alters the IC constraints that other buyers face in other submarkets, thus affecting the set of feasible contracts that those buyers can offer to attract other types of sellers. This, in turn, will affect the payoff of other sellers in other submarkets. The buyers in the market economy does not take this effect into account. The planner, in contrast, internalizes these externalities and therefore is able to increase welfare.

The inability of buyers to internalize the externalities they create on others in equilibrium is similar to the situation in random search models with ex-post bargaining (like Mortensen and Pissarides (1994)) in which the share of the surplus that buyers get is exogenously fixed, so the outcome is generally inefficient. Here, although the division of the surplus to buyers and sellers is not exogenously fixed, it is endogenously pinned down by two constraints that IC and free entry impose on the allocation, so it is generally unlikely that the efficiency is achieved. The planner can internalize these externalities, because he is not constrained by the free entry condition at each submarket, so he can make the buyers' share of the surplus satisfy Hosios condition.

4.2 What If There Are No Gains from Trade for Some Types?

Guerrieri et al. (2010) show that in the asset market with lemons, if there are no gains from trade only for type one, that is, $b_1 - c_1 < 0$ and $b_2 - c_2 > 0$, then the entire market will shut down. I show that in this case, it is not guaranteed that the planner can help: The planner can improve the welfare only if the measure of type two sellers in the population is sufficiently high. (See the appendix.) In other words, if the distortion is so severe that other types become inactive, then the planner might not be able to help.

The intuition is that if type two is inactive in the equilibrium, it means that given IC of type one, the highest payoff that type two can get in the market is negative, so type two choose not to participate in the market. If π_2 is not sufficiently high, then the amount of resources needed to subsidize type one is more than the surplus that type two sellers can generate. In this case, the planner cannot increase welfare.

	Complete information	Equilibrium allocation	Planner's allocation
θ_1	θ_1^{CI}	θ_1^{CI}	θ_1^{CI}
θ_2	θ_2^{CI}	$\theta_2 > \theta_2^{CI}$	θ_2^{CI}
h_1	h_1^{CI}	h_1^{CI}	h_1^{CI}
h_2	h_2^{CI}	$h_2 > h_2^{CI}$	h_2^{CI}
p_1	$p_1^{CI} \equiv v_1(h_1^{CI}) - \frac{k}{q(\theta_1^{CI})}$	$v_1(h_1^{CI}) - \frac{k}{q(\theta_1^{CI})}$	$p_1^{CI} - t_{m,1}$
p_2	$p_2^{CI} \equiv v_2(h_2^{CI}) - \frac{k}{q(\theta_2^{CI})}$	$v_2(h_2) - \frac{k}{q(\theta_2)}$	$p_2^{CI} - t_{m,2}$
$t_{m,1}$	0	0	$-\frac{\pi_2}{m(\theta_1^{CI})}(U_2^{CI} - U_1^{CI}) + \pi_2(\tau - 1)\phi_2(h_1^{CI})$
$t_{m,2}$	0	0	$\frac{\pi_1}{m(\theta_2^{CI})}(U_2^{CI} - U_1^{CI}) - \pi_1(\tau - 1)\frac{m(\theta_1^{CI})}{m(\theta_2^{CI})}\phi_2(h_1^{CI})$

Table 2: The rat race results.

5 Example 2: The Rat Race

In this section I study another example from Guerrieri et al. (2010), the rat race, which was originally discussed in Akerlof (1976). The main point of this example is that the complete information allocation is achievable only through a separating allocation, in contrast to the previous example (asset market with lemons) where the complete information allocation was achievable through a pooling allocation (if $\pi_1 b_1 + \pi_2 b_2 \geq c_2$). The planner here achieves the complete information allocation by separating different types and using appropriate transfers.

There are two types of workers (as sellers) on one side and firms (as buyers) on the other side of the market. The payoff of a type i worker matched with a firm from h hours of work and consuming t units of the numeraire good is $t - \phi_i(h)$. The worker's payoff is 0 if unmatched. The payoff of a firm matched with a type i worker when the worker works for h hours and the firm pays t units of the numeraire good (either to the worker or to the planner) is $v_i(h) - t - k$. The firm's payoff is $-k$ if unmatched. As a special case of the original setting, here $I = 2$, $n = 1$, $a \equiv h$ and $u_i(h) = -\phi_i(h)$. Matching function $m(\theta)$ is strictly concave and twice differentiable. I make the following assumptions on the payoff functions of the agents:

Assumption 4. *In the rat race example,*

1. *ϕ_i is differentiable, increasing, strictly convex and $\phi_i(0) = \phi_i'(0) = 0$.*

2. *For all h, $\phi_1(h) = \tau\phi_2(h)$ where $\tau > 1$.*

3. *v_i is differentiable, increasing and strictly concave.*

4. *For all h, $v_1(h) \leq v_2(h)$ and $v_1'(h) \leq v_2'(h)$.*

Assumption 2 was sufficient for Proposition 2. I argue here that under Assumption 4, the requirement of the Proposition 2 is satisfied and therefore I can use that result to characterize the constrained efficient allocation for the rat race. Part 1 of Assumption 2 is satisfied because $\tau > 1$. Part 2 of Assumption 2 (single crossing of $u(.)$) is satisfied because $-\phi_2(h) - (-\phi_1(h))$ is increasing in h. Part 3 of Assumption 2 (single crossing of the surplus) is satisfied because $v_2(h) - \phi_2(h) - (v_1(h) - \phi_1(h))$ is increasing in h. Part 4 of Assumption 2 (supermodularity of the surplus) is satisfied because h is just one dimensional. Part 5(a) of Assumption 2 (monotonicity of the surplus) is satisfied because $v_2(h) - \phi_2(h) \geq v_1(h) - \phi_1(h)$.

Proposition 4. *If Assumption 4 holds and $U_i^{CI} > 0$ for all i, then the planner achieves the complete information allocation. See the fourth column of Table 2 for the full description of the planner's allocation.*

This result is a special case of Proposition 2. The planner subsidizes type one ($p_1 > p_1^{CI}$) and taxes type two ($p_2 < p_2^{CI}$) to achieve efficiency. By offering this scheme of transfers, allocating the low type workers higher wage and the high type workers lower wage than their wages in the equilibrium under complete information, the planner tries to discourage low type workers from applying to submarket two, thus reducing the cost of private information.

An interesting thing about this result is that the planner achieves the complete information allocation regardless of the distribution of types. The intuition is that here the planner can make positive amount of money over *each* submarket, when he sets payments such that type one gets 0 payoff.

Guerrieri et al. (2010) make the same assumptions except that they they do not require $v_1'(h) \leq v_2'(h)$. When $U_2^{CI} - U_1^{CI} \geq (\tau - 1)m(\theta_2^{CI})\phi_2(h_2^{CI})$, then the equilibrium does not achieve the complete information allocation. They propose a pooling allocation which Pareto dominates the equilibrium allocation if π_1 is sufficiently small, although the pooling allocation does not achieve the complete information allocation. As stated earlier, the planner can achieve the complete information allocation regardless of π_1. Moreover, if π_1 is sufficiently small, then the planner's allocation Pareto dominates the equilibrium allocation.

6 Extension: Asset Market with Continuous Type Space

The original setting in Section 2 has a discrete type space. In this section, I extend the analysis to a continuous type space. This extension is interesting, because it makes it possible for us to consider cases in which the value of assets to sellers does not have the same order as the value of assets to buyers. I could do the same thing with a discrete type space with more than two types, but the analysis with a continuous type space is simpler. Also, studying this case makes it possible for us to compare the planner's allocation with the equilibrium allocation in Chang (2012) (and somewhat with Guerrieri and Shimer (2014)). Although Chang has a dynamic environment, but the main ideas are captured in our static case[11]. Since this environment is not a special case of the original setting in Section 2, I need to define the constrained efficient allocation again. The mains ideas discussed so far go through for this case as well, but the mathematical tools that I use to characterize the planner's allocation are different.

6.1 Environment

There is a continuum of measure one of heterogeneous sellers indexed by $z \in Z \equiv [z_L, z_H] \subset \mathbb{R}$, with $F(z)$ denoting the measure of sellers with types below z. F is continuously differentiable and strictly increasing in z and F' is its derivative (a density function). Type z is seller's private information. Similar to the original setting, buyers' and sellers' payoffs are quasi-linear. A buyer's payoff who enters the market and gets matched with a type z is $h(z) - t - k$ where t denotes the amount of a numeraire good that he produces and $h(z)$ is the value of the asset to the buyer in terms of the numeraire good. His payoff is $-k$ if unmatched. The payoff of a type z seller matched with a buyer is $t - c(z)$ where t denotes the amount of the numeraire good that he consumes and $c(z)$ is the value of the asset to the seller in terms of the muneraire good. His payoff is 0 if unmatched. Functions $h : Z \to R$ and $c : Z \to R$ are twice continuously differentiable. Matching function $m(.)$ is increasing, strictly concave and twice differentiable with strictly decreasing elasticity.

6.2 Complete Information Case

Here, I mostly follow the discussion of the complete information case for the discrete type in Section 3. Consider the market economy with the complete information. The buyers who

[11] In the dynamic setting, the planner has some inter-temporal considerations, because the distribution of types in the population does not necessarily remain the same over time, because some types get matched more quickly than others and exit the market. This fact raises a new and interesting tradeoff, whether the planner wants to get rid of low types early or he wants to have all types together all the way to the end. The analysis of the dynamic setting is beyond the scope of this paper. Since the equilibrium allocation is distribution free, the equilibrium analysis is much easier than the planner's analysis in the dynamic case. If I assume in the dynamic setting that when sellers sell their assets, they are endowed a new asset with the same quality, the same results can be obtained from the dynamic setting as in the static setting, because the distribution of types does not change over time.

attempt to attract type z sellers solve the following problem:

$$U^{CI}(z) = \max_{\theta,p}\{m(\theta)(p - c(z))\}$$
$$\text{s.t. } q(\theta)(h(z) - p) = k.$$

Let $\theta^{CI}(z)$ and $p^{CI}(z)$ denote the market tightness and the price that solve this problem. I make the assumption that $U^{CI}(z) > 0$ for all z, that is, that there are positive gains from trade for all types. Similar to the discrete type case, $U^{CI}(z) = \max_\theta\{m(\theta)(h(z)-c(z))-k\theta\}$. Also $\theta^{CI}(z)$ solves

$$m'(\theta)(h(z) - c(z)) = k, \tag{1}$$

for both the planner and the market economy when there is complete information. The left hand side of equation 1 is the marginal benefit of adding one more buyer to the submarket composed of z sellers. The right hand side is the marginal cost of doing that. The planner keeps adding buyers to each submarket until the marginal cost and marginal benefit become equal. I calculate the share of the surplus that sellers get in equilibrium to verify that the Hosios condition (Hosios (1990)) is satisfied for that:

$$\frac{p^{CI}(z) - c(z)}{h(z) - c(z)} = \frac{U^{CI}(z)}{m(\theta^{CI})(h(z) - c(z))} = \frac{m(\theta^{CI})(h(z) - c(z)) - k\theta^{CI}}{m(\theta^{CI})(h(z) - c(z))}$$
$$= \frac{m(\theta^{CI})(h(z) - c(z)) - \theta^{CI}m'(\theta^{CI})(h(z) - c(z))}{m(\theta^{CI})(h(z) - c(z))} = -\frac{\theta^{CI}q'(\theta^{CI})}{q(\theta^{CI})} \equiv \eta(\theta^{CI}),$$

where the third equality follows equation 1. The share of the surplus that type z sellers get from the match is $\frac{p-c(z)}{h(z)-c(z)}$. Hosios condition states that a necessary condition for efficiency of the allocation is that this share must be equal to the elasticity of matching function with respect to the number of sellers. The equilibrium allocation under complete information satisfies this property.

6.3 Definition of Planner's Problem

In order to define the planner's problem, I need to introduce new notations. Let $\phi \in \Phi \in \mathbb{R}$ denote the index of submarkets. Denote by $G(\phi)$ the measure of buyers in submarkets with indices less than ϕ. Denote by $H(\phi, z)$ the measure of sellers with types below z in submarkets with indices less than ϕ. The fraction of sellers in submarkets with indices less than ϕ is denoted by $H_\Phi(\phi)$. The fraction of sellers with types below z is denoted by $H_Z(z)$. I impose the restriction that H_Φ is absolutely continuous with respect to G. This restriction means that if ϕ is not in the support of G, then no seller can enter that submarket. Now, I define the allocation for the continuous type space.

Definition 4. *An allocation is a set which consists of distributions G and H and functions $p : \Phi \to \mathbb{R}$ and $t_m : \Phi \to \mathbb{R}$.*

Denote by $p(\phi)$ the amount of the numeraire good to be transferred from buyers to sellers conditional on trade in submarket ϕ. Denote by $t_m(\phi)$ the amount of the numeraire good to be transferred from buyers to the planner in submarket ϕ conditional on trade. As stated earlier, I assume throughout this section that there are positive gains from trade for all types. Therefore, I assume without loss of generality that the amount of the numeraire good that the planner allocates to the sellers who don't apply to any submarket is 0 without loss of generality. Therefore, there is no T as a part of description of an allocation in this section.

Definition 5. *An allocation $\{G, H, p, t_m\}$ is feasible if it satisfies the following conditions:*

1. *(Sellers' maximization) $(\phi, z) \in$ supp H only if*

$$\phi \in \arg \max_{\phi' \in \text{ supp } G} \{m(\theta_{GH}(\phi'))(p(\phi') - c(z))\} \text{ and } m(\theta_{GH}(\phi))(p(\phi) - c(z)) \geq 0.$$

2. *(Buyers' zero payoff) For any $\phi \in$ supp G,*

$$q(\theta_{GH}(\phi))\big[\int_Z h(z)\theta_{GH}(\phi)\frac{dH}{dG} - p(\phi) - t_m(\phi)\big] = k,$$

3. *(Planner's budget constraint)*

$$\int_{\text{supp } G} q(\theta_{GH}(\phi))t_m(\phi)dG \geq 0,$$

where $\theta_{GH}(\phi) = (\frac{dH_\Phi}{dG})^{-1}$ (Radon-Nikodym derivative), and $H_Z(z) = F(z)$.

The conditions above are similar to the conditions in the definition of feasible allocation in the original setting of Section 2. The first condition states that if a type z seller chooses submarket ϕ, then this submarket must maximize his payoff among all open submarkets and must deliver him a positive payoff. The second condition states that every open submarket must yield exactly zero profit to the buyers who choose that submarket. The last condition states that the net amount of transfers that buyers make to the planner must be non-negative. Market tightness at every submarket is given by the inverse of Radon-Nikodym derivative of H_Φ with respect to G. Finally, $H_Z(z) = F(z)$ states that the aggregate measure of sellers with types below z in all submarkets must exactly equal to the measure of sellers with types below z in the population.

Definition 6. *A constrained efficient allocation is a feasible allocation* $\{G, H, p, t_m, T\}$ *which maximizes welfare among all feasible allocations:*

$$\max \int_{supp\ H} \left[m(\theta_{GH}(\phi))(h(z) - c(z)) - k\theta_{GH}(\phi) \right] dH$$

$$\textit{Subject to: } \{G, H, t_s, t_m\} \textit{ is feasible.}$$

6.4 Results

In order to solve the planner's problem, I use somewhat a backward approach. I first guess that the planner can achieve the complete information allocation. That is, the planner can maximize his objective function for each type separately. Then I find a set of transfers, $p(.)$ and $t_m(z)$, such that sellers' maximization condition and buyers' zero profit condition are satisfied. Given these schemes of payments, I derive sufficient conditions under which the planner's budget condition is satisfied.

Assumption 5. *For all* z, $c'(z) > 0$ *and either*

1. $h'(z) \leq 0$, *or*
2. $h'(z) \leq c'(z)$ *and* $\psi(\frac{k}{h(z)-c(z)})\left[\frac{h(z)-c(z)}{c'(z)}\right] \geq \frac{F(z)}{F'(z)}$, *where* $\psi(.) \equiv \eta(m'^{-1}(.))$ *and* $\eta(\theta) \equiv -\frac{\theta q'(\theta)}{q(\theta)}$.

Proposition 5. *If Assumption 5 holds and* $U^{CI}(z) > 0$ *for all* z*, different types trade in different submarkets (so without loss of generality submarkets can be indexed by type* z*). Then the planner achieves the complete information allocation, that is,*

$$\theta(z) = \theta^{CI}(z), \textit{ for all } z.$$

Also, the payments at each submarket is given by:

$$p(z) = c(z) + \frac{U(z_H) + \int_z^{z_H} m(\theta(z_0))c'(z_0)dz_0}{m(\theta(z))},$$

$$t_m(z) = h(z) - p(z) - \frac{k}{q(\theta(z))}.$$

where $U(z_H) = \int \left[m(\theta(z))(h(z) - c(z)) - k\theta(z) - m(\theta(z))c'(z)\frac{F(z)}{F'(z)} \right] dF(z)$.

To have a rough idea how the planner can take care of private information, let's write the IC problem. I assume (without loss of generality) that sellers are allocated to different submarkets through a direct mechanism: If type z reports $\hat{z}$, his payoff is given by $m(\theta(\hat{z}))(p(\hat{z})-c(z))$. The agent chooses a $\hat{z}$ which maximizes his payoff:

$$\max_{\hat{z}}\{m(\theta(\hat{z}))(p(\hat{z})-c(z))\}. \tag{2}$$

I use the assumption that $c'(z)>0$ throughout this section, so single crossing condition holds. As already discussed in Section 2, $\theta(z)$ being decreasing and $c(z)$ being increasing in z together imply that there exists a set of transfers that satisfies IC. Now assume $h'(z)\leq 0$ for all z or $h'(z)\leq c'(z)$ for all z. In either case, $\theta^{CI}(z)$ is decreasing in z according to equation 1. Therefore, such transfers exist. I provide sufficient conditions in the proof such that the planner's budget constraint holds. If $h'(z)\leq 0$, the planner has enough resources to distribute among agents regardless of the distribution. If $h'(z)\leq 0$ is not satisfied for some z but $h'(z)-c'(z)\leq 0$ still holds, I need another condition which relates the distribution of types to the payoff and matching functions (second part of Assumption 5) to ensure that the planner's budget constraint is satisfied.

In the next proposition, I keep the assumption that $c'(z)>0$ and $h'(z)-c'(z)\leq 0$, but the distribution of types is such that the planner cannot achieve the complete information allocation. Therefore, the probability of matching for almost all types must be distorted (relative to the complete information allocation) so that IC and budget constraint are both satisfied.

Assumption 6. *For all z, $h(z)-c(z)-c'(z)\frac{F(z)}{F'(z)}>0$ and $\frac{d}{dz}\left[h(z)-c(z)-c'(z)\frac{F(z)}{F'(z)}\right]\leq 0$.*

Proposition 6. *Assume $c'(z)>0$, $h'(z)-c'(z)\leq 0$ and $U^{CI}(z)>00$ for all z. Also, suppose Assumption 6 holds. If the complete information allocation is not achievable, then there exists a $\mu>0$ such that the market tightness $\theta(.)$ for the constrained efficient allocation solves the following equations:*

$$m'(\theta(z))\left[h(z)-c(z)-\frac{\mu}{1+\mu}c'(z)\frac{F(z)}{F'(z)}\right]=k, \tag{3}$$

$$\int\left[m(\theta(z))\left[h(z)-c(z)-c'(z)\frac{F(z)}{F'(z)}\right]-k\theta(z)\right]F'(z)dz=0.$$

Moreover, $p(z)$ and $t_m(z)$ are obtained similarly as before:

$$p(z)=c(z)+\frac{U(z_H)+\int_z^{z_H}m(\theta(z_0))c'(z_0)dz_0}{m(\theta(z))},$$

$$t_m(z) = h(z) - p(z) - \frac{k}{q(\theta(z))},$$

and $U(z_H) = 0$.

Proposition 5 requires that the virtual surplus of each type, $m(\theta(z))[h(z) - c(z) - c'(z)\frac{F(z)}{F'(z)}] - k\theta(z)$, be positive (if $h'(z) \leq 0$ is not satisfied for some z). However, if the virtual surplus of some types is negative, then Proposition 6 requires that at least $h(z) - c(z) - c'(z)\frac{F(z)}{F'(z)}$ be positive for all z. Also this proposition requires $h(z) - c(z) - c'(z)\frac{F(z)}{F'(z)}$ to be decreasing in z to ensure that the monotonicity constraint is satisfied. Note that generally $t_m(z) \neq 0$ for almost all types. The point is that although the complete information allocation is not achievable under the premises of this proposition, the planner effectively uses transfers to achieve higher welfare than the equilibrium. The intuition is the same as in the simple two type example.

6.5 Equilibrium Allocation

I report the results of a static version of Chang (2012) here and compare equilibrium allocation with the planner's one. Her model is dynamic, but my model is static. See footnote 12 why I consider a static model. Chang assumes that utility of holding the asset until finding a buyer is different across different types of assets. Similarly, I assume sellers with high z values their assets more ($c'(.) > 0$).

Assumption 7. $c'(z) > 0$, $h'(z) \geq 0$ *for all* z.

Proposition 7 (From Chang (2012))**.** *If Assumption 7 holds and if* $U^{CI}(z) > 00$ *for all* z, *then there exists a unique equilibrium. The equilibrium is separating. The market tightness solves the differential equation 5. The initial condition is given by* $\theta(z_L) = \theta^{CI}(z_L)$. *Prices are given by* $p(z) = h(z) - \frac{k}{q(\theta(z))}$.

See her paper for the formal proof. I just explain the logic behind her result. First note that the IC constraints that agents face in the market economy is the same as ones that the planner faces, therefore I can use equation 2 to describe IC constraints in the equilibrium. The only difference is that the price is different in the market economy, because it is pinned down by the free entry condition. In the market economy, she shows that any equilibrium under Assumption 7 is separating, so free entry implies that $p(z) = h(z) - \frac{k}{q(\theta(z))}$ for all z. Therefore, the payoff of type z in the market economy, denoted by $U^{eq}(z)$, is calculated as follows:

$$U^{eq}(z) = \max_{\hat{z}}\{m(\theta(\hat{z}))(h(\hat{z}) - c(z)) - k\theta(\hat{z})\}, \tag{4}$$

Condition	Necessary condition for I.C	Initial conditions
$0 < c'(z)$ and $0 < h'(z)$	$\frac{d\theta}{dz} < 0$	$\theta(z_L) = \theta^{CI}(z_L)$
$0 < c'(z)$ and $0 > h'(z)$	$\frac{d\theta}{dz} < 0$	$\theta(z_H) = \theta^{CI}(z_H)$
$0 > c'(z)$ and $0 < h'(z)$	$\frac{d\theta}{dz} > 0$	$\theta(z_L) = \theta^{CI}(z_L)$
$0 > c'(z)$ and $0 > h'(z)$	$\frac{d\theta}{dz} > 0$	$\theta(z_H) = \theta^{CI}(z_H)$

Table 3: Equilibrium allocation in different cases in the asset market with continuous type space

where the objective function is the payoff of type z if he reports type $\hat{z}$. FOC with respect to $\hat{z}$ (together with the assumption of differentiability of $\theta(z)$) yields the following equation:

$$[m'(\theta(z))(h(z) - c(z)) - k]\frac{d\theta}{dz} + m(\theta(z))h'(z) = 0, \tag{5}$$

where I used the fact that at the solution, $\hat{z} = z$ due to IC. With respect to the initial condition, roughly speaking, the market delivers the complete information payoff to the type which has the most incentive to deviate. For example, when $h'(.) \geq 0$, the lowest type has the most incentive to deviate, so his allocation is set to the complete information level. The necessary and sufficient condition for IC and the initial conditions are depicted in Table 3 under different assumptions, for example if $c' > 0$ and $h' \leq 0$. I maintain this assumption that there are positive gains from trade for all types.

6.6 Two-dimensional Private Information

Chang (2012) assumes in another part of her paper that sellers have another dimension of private information. Some sellers get liquidity shocks so they need to sell their assets quickly. What is relevant to our discussion is that following this extension, it is possible that function h has a local maximum (keeping the assumption $c'(z) > 0$ fixed). If so, she proves that full separation in the market is not possible. Also, she derives some conditions under which an equilibrium which resembles fire-sales exists, where many low type sellers and some high type sellers who need liquidity sell their assets with a lower price but very quickly. My characterization, in contrast, shows if $h'(z) \leq c'(z)$ and if part 2 of Assumption monotonicity assumption in continuous case or Assumption 6 holds, even if h has a local maximum, then the constrained efficient allocation is separating, that is, the planner wants different types to trade in different submarkets. This case is depicted in Figure 3 where $h'(z)$ is drawn in terms of $c'(z)$ for all z.

Now suppose $h'(z) - c'(z) \leq 0$ is violated for some z. For example, $h - c$ has one local minimum, but $h'(z) \geq 0$ and $c'(z) > 0$, as depicted in Figure 4. The equilibrium in this case is separating. The planner's allocation, in contrast, involves some pooling, because monotonicity constraint (that $\theta(z)$ should be decreasing in z) cannot be satisfied through any separating allocation[12]. The bottom line is that pooling of types occurs under different conditions in the planner's allocation and the equilibrium allocation.[13] The welfare under planner's allocation is strictly higher by the same argument as in Proposition 1 even if the planner does not achieve the complete information allocation.

7 Conclusion

I have characterized the constrained efficient allocation in an environment with directed search and private information. Under similar assumptions that Guerrieri et al. (2010) characterize the unique equilibrium, the planner can achieve strictly higher welfare than the equilibrium if the equilibrium fails to achieve the complete information allocation. Under a different assumption (Assumption 2), the planner can even achieve the complete information allocation. The main idea is that the planner uses transfers rather than market tightness or production level to have IC satisfied as long as agents' participation constraints and the planner's budget constraint are satisfied. In the market economy, the buyers do not take into account the effect of their entry on the set of feasible submarkets available to buyers who want to attract other types of sellers. This is, entry of a buyer to a submarket changes the payoff of sellers in that submarket and this in turn, through incentive compatibility, changes what buyers can post in other submarkets and eventually changes the payoff of sellers in other submarkets. The planner takes this externality into account and therefore, he is able to increase welfare by imposing appropriate taxes and subsidies.

I illustrated my results in different examples like asset markets and labor markets. In an asset market example in Section 6, I showed that if the value of assets to sellers is increasing and the surplus created by assets is decreasing in the type of assets ($c'(.) > 0$

[12]Solving explicitly for the planner's allocation in this case does not give us new insights, so I skip its analysis. The method used in this situation is called bunching and is discussed in mechanism design literature. For example, see the appendix of chapter 7 in Fudenberg and Tirole (1991).

[13]Roughly speaking, the planner is concerned with the surplus from the match not the value of the match to buyers only, so in the conditions regarding the planner's allocation, usually $h - c$ shows up. The buyers in the equilibrium are concerned with the value of the assets to themselves, so in the conditions regarding the allocation, usually h shows up, not h separately.

and $h'(.) - c'(.) \leq 0$), then the planner can achieve the complete information allocation by subsidizing low types and taxing high types. I compared my results with Chang (2012) and showed that the conditions under which the equilibrium allocation and planner's allocation are separating are different.

I have assumed in this paper that agents match bilaterally. An important question is that if one considers a more general case and allows several sellers to meet with a buyer so sellers face some competition after meeting a buyer, whether it induces sellers to reveal their types less costly? Also, does the equilibrium remain constrained inefficient? In a work in progress, I address these questions and try to characterize both equilibrium and constrained efficient allocation in such an environment. My conjecture is that although the equilibrium might be more efficient than the case with bilateral meetings, the equilibrium will remain constrained inefficient. This is yet to be verified.

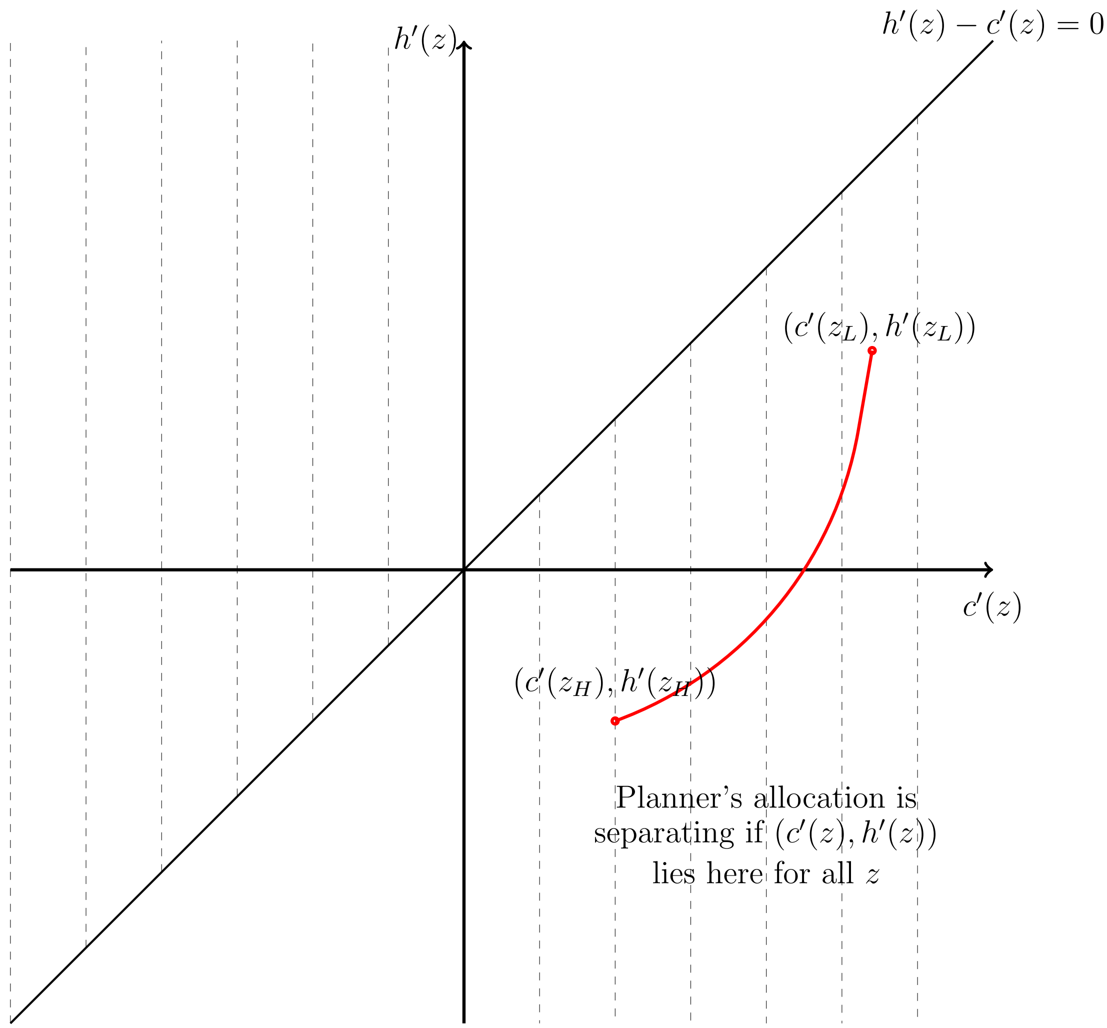

Figure 3: Assume that $h(z)$ has a local maximum but $h'(z) - c'(z) \leq 0$ for all z. Also assume that the distribution is such that part 2 of Assumption 5 or Assumption 6 holds. Because the value of assets with higher z to sellers is not monotone in z, the equilibrium will involve some pooling. Chang (2012) shows this point formally in her Proposition 5. However, the planner's allocation is separating. The planner can actually achieve the complete information allocation according to Proposition 5. Symmetrically, if $c'(z) < 0$ and $h'(z) - c'(z) \geq 0$, and if a similar condition to part 2 of Assumption 5 or Assumption 6 holds, then the planner will get a separating allocation.

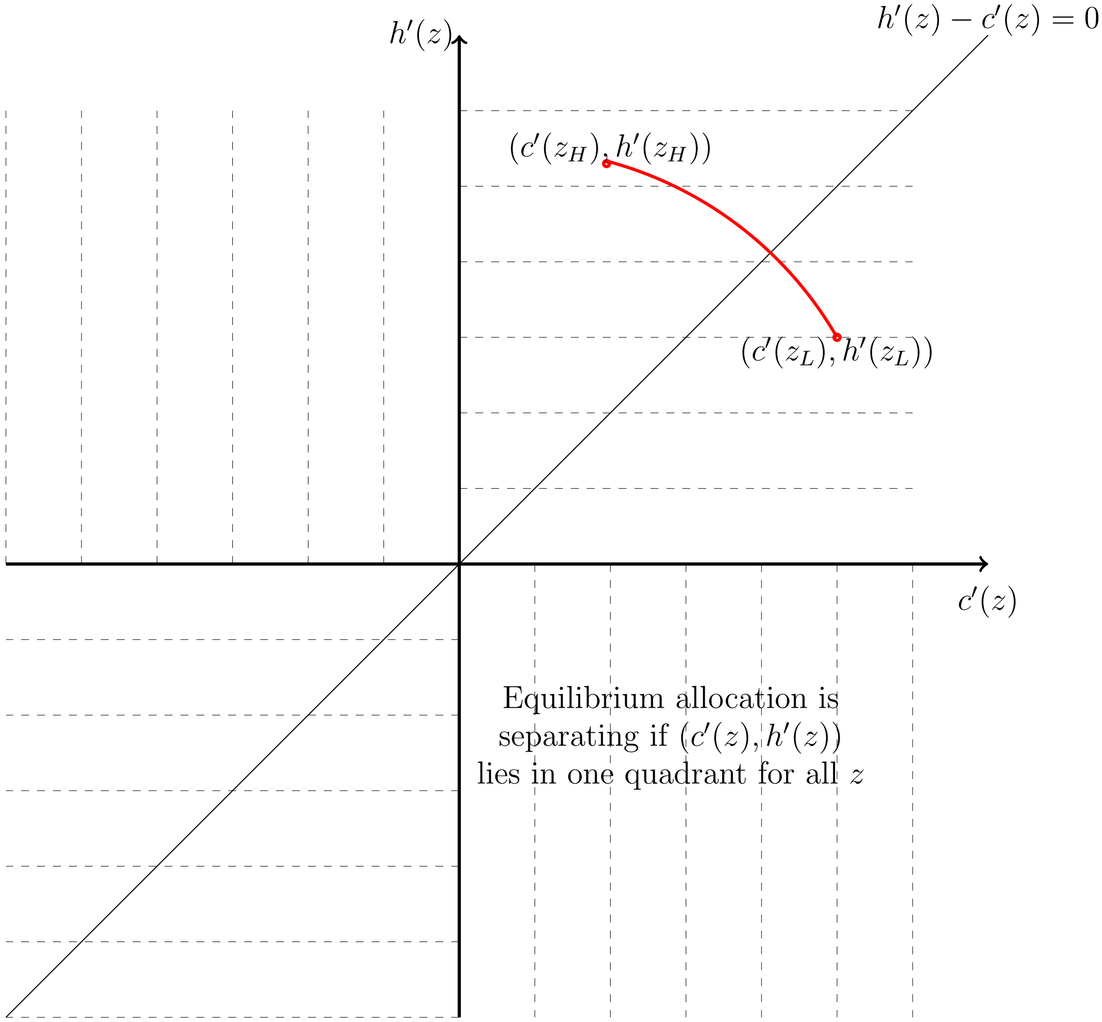

Figure 4: Assume that $h - c$ has one local minimum and $h'(z) \geq 0$ for all z. Since $h'(z) \geq 0$ for all z, then the equilibrium allocation is separating. However, the planner's allocation is pooling, because monotonicity constraint is not satisfied. Indeed, the planner wants to pool all types higher than a threshold in one submarket.

8 Appendix

8.1 Proof of Result 1

Proof of result 1. I begin from equilibrium allocation and modify it to improve the welfare. Consider a type i seller who gets a strictly positive payoff in the equilibrium and does not produce a_i^{CI} or does not get matched with probability $m(\theta_i^{CI})$. Such a type exists, because the equilibrium fails to achieve the complete information allocation. Let $\tilde{i}$ denote this type. I define a set of problems, similar but not the same as one in Guerrieri et al. (2010), and characterize its solution. From that solution, I construct an allocation and show that the constructed allocation is feasible and yields higher welfare for the planner than the equilibrium allocation.

First, according to proposition 3 in their paper, Guerrieri et al. (2010) show that the following set of problems characterizes the equilibrium.

Problem 1 ($P_i(0)$)**.**

$$\max_{\theta\in[0,\infty],(a,p)\in\bar{Y}}\{m(\theta)(u_i(a)+p)\}$$

subject to

$$q(\theta)(v_i(a)-p)\geq k$$

and

$$m(\theta)(u_j(a)+p)\leq \bar{U}_j(0) \textit{ for all } j<i.$$

More precisely, define problem $P(0)$ to be the set of problems $P_i(0)$ for all i. Let $\bar{U}_i(0)$ be the value of the objective function in problem $P_i(0)$ given ($\bar{U}_1(0)$,$\bar{U}_2(0)$,...,$\bar{U}_{i-1}(0)$) if $\bar{U}_i(0)$ is strictly greater than 0 and $\bar{U}_i(0)=0$ otherwise. Denote by $I^*(0)\subseteq\{1,2,...,I\}$ the set of types such that the constraint set in $P_i(0)$ is non-empty and $\bar{U}_i(0)$ is strictly greater than 0. For any $i\in I^*(0)$, let $(\bar{\theta}_i(0),\bar{a}_i(0),\bar{p}_i(0))$ denote the solution to problem $P_i(0)$ given ($\bar{U}_1(0)$,$\bar{U}_2(0)$,...,$\bar{U}_{i-1}(0)$). Now, consider another set of problems which is basically a perturbation of the above problem is a specific way:

Problem 2 ($P_i(\epsilon),\epsilon>0$)**.**

$$\max_{\theta\in[0,\infty],(a,p)\in\bar{Y}}\{m(\theta)(u_i(a)+p)+\delta_i\}$$

subject to

$$q(\theta)(v_i(a)-p)\geq k,$$

and

$$m(\theta)(u_j(a)+p)+\delta_i \leq \bar{U}_j \text{ for all } j<i,$$

where $\delta_i=\epsilon$ *if* $i<\tilde{i}$ *or* $i \notin I^*(0)$ *and* $\delta_i=0$ *otherwise.*

Similarly as above, define problem $P(\epsilon)$ to be the set of problems $P_i(\epsilon)$ for all i. Let $\bar{U}_i(\epsilon)$ be the value of the objective function in problem $P_i(\epsilon)$ given ($\bar{U}_1(\epsilon)$,$\bar{U}_2(\epsilon)$,...,$\bar{U}_{i-1}(\epsilon)$) if $\bar{U}_i(\epsilon)$ is strictly greater than ϵ and $\bar{U}_i(\epsilon)=\epsilon$ otherwise. Denote by $I^*(\epsilon) \subseteq \{1,2,...,I\}$ the set of types such that the constraint in $P_i(\epsilon)$ is non-empty and $\bar{U}_i(\epsilon)$ is strictly greater than ϵ. For any $i \in I^*(\epsilon)$, let $(\bar{\theta}_i(\epsilon), \bar{a}_i(\epsilon), \bar{p}_i(\epsilon))$ denote the solution to problem $P_i(\epsilon)$ given ($\bar{U}_1(\epsilon)$,$\bar{U}_2(\epsilon)$,...,$\bar{U}_{i-1}(\epsilon)$).

I begin from the equilibrium allocation and construct another allocation which leads to strictly higher welfare. As stated earlier, type $\tilde{i}$ does not achieve the complete information allocation, therefore, some constraints in problem $P_{\tilde{i}}(0)$ must be binding, so that the value of the objective function is strictly less than that under complete information (where there are no incentive compatibility constraints).

In the allocation proposed below ,$\{Y^P, \lambda, \theta, \Gamma, T\}$, the planner relaxes the constraints of problem $P_{\tilde{i}}(0)$ without distorting the allocation for lower types or inactive types.

$$y_i = \begin{cases} (\bar{a}_i(0), \bar{p}_i(0)+\frac{\epsilon}{m(\bar{\theta}_i(0))}, 0, 0) & \text{if } 1 \leq i < \tilde{i} \text{ and } i \in I^*(0) \\ (\bar{a}_i(\epsilon), \bar{p}_i(\epsilon), 0, 0) & \text{if } \tilde{i} \leq i \leq I \text{ and } i \in I^*(0). \end{cases} \tag{6}$$

$$Y^P = \{y_i\}_{i \in I^*(0)}, \theta(y_i)=\bar{\theta}_i(\epsilon), \gamma_i(y_i)=1, \lambda(\{y_i\})=\bar{\theta}_i(\epsilon)\pi_i \text{ and } T=\epsilon. \tag{7}$$

I suppress the index ϵ for this allocation to reduce the notation. (Instead of $Y^P(\epsilon)$, I use Y^P). Since the constraint of $P_i(\epsilon)$ is exactly the same as $P_i(0)$ for types below $\tilde{i}$, they get exactly $\bar{U}_i(\epsilon)=\bar{U}_i(0)+\epsilon$. For types above $\tilde{i}$, if they are not active in the equilibrium, that is, $i>\tilde{i}$ and $i \notin I^*(0)$, then $\bar{U}_i(\epsilon)=\bar{U}_i(0)+\epsilon=\epsilon$.

For type $\tilde{i}$, problem $P_i(\epsilon)$ maximizes the objective function given ($\bar{U}_1(\epsilon)$,$\bar{U}_2(\epsilon)$,...,$\bar{U}_{\tilde{i}-1}(\epsilon)$). Since some constraints were binding at $P_{\tilde{i}}(0)$, problem $P_{\tilde{i}}(\epsilon)$ yields to strictly higher value of the objective function, because those constraint are now relaxed.

For types above $\tilde{i}$ who are active in the equilibrium, the objective function is weakly higher than the equilibrium allocation. I show the latter claim by induction. Fix $\tilde{i}<j$ and assume that for all k such that $\tilde{i} \leq k < j$, then $\bar{U}_k(\epsilon) \geq \bar{U}_k(0)$, that is, the value of the objective functions is higher than that in the equilibrium. This implies that the constraint set in problem $P_j(\epsilon)$ is bigger for all k with $\tilde{i} \leq k < j$. For $k<\tilde{i}$, we already know that the constraint set is bigger by definition of $P_k(\epsilon)$. Hence, the value of the objective function in

$P_j(\epsilon)$ should be weakly higher than that in equilibrium. To summarize, so far I have proved the following:

$$\bar{U}_i(\epsilon) = \bar{U}_i(0) + \epsilon \text{ for all } i < \tilde{i} \text{ or } i \notin I^*(0),$$

$$\bar{U}_{\tilde{i}}(\epsilon) > \bar{U}_{\tilde{i}}(0),$$

$$\text{and } \bar{U}_i(\epsilon) \geq \bar{U}_i(0) \text{ for all } i > \tilde{i} \text{ and } i \in I^*(0). \tag{8}$$

Under allocation $\{Y^P, \lambda, \theta, \Gamma, T\}$, the planner needs to raise $\tilde{\epsilon} \equiv \epsilon(\sum_{i=1}^{\tilde{i}-1} \pi_i + \sum_{i=\tilde{i}+1, i \notin I^*(0)}^{I} \pi_i)$ units of the numeraire good from the agents to have the budget constraint satisfied. Consider the following allocation, $\{Y^{P\prime}, \lambda', \theta', \Gamma', T'\}$, which is derived from allocation $\{Y^P, \lambda, \theta, \Gamma, T\}$ by redistributing $\tilde{\epsilon}$ units of the numeraire good across sellers such that incentives are not affected.

$$y_i' = \begin{cases} (\bar{a}_i, \bar{p}_i + \frac{\epsilon - \tilde{\epsilon}}{m(\bar{\theta}_i(0))}, -\frac{\epsilon - \tilde{\epsilon}}{m(\bar{\theta}_i(0))}) & \text{if } 1 \leq i < \tilde{i} \text{ and } i \in I^*(0) \\ (\bar{a}_i, \bar{p}_i - \frac{\tilde{\epsilon}}{m(\bar{\theta}_i(\epsilon))}, \frac{\tilde{\epsilon}}{m(\bar{\theta}_i(\epsilon))}) & \text{if } \tilde{i} \leq i \leq I \text{ and } i \in I^*(0). \end{cases} \tag{9}$$

$$Y^{P\prime} = \{y_i'\}_{i \in I^*(0)}, \theta'(y_i') = \bar{\theta}_i(\epsilon), \gamma_i'(y_i') = 1, \lambda'(\{y_i'\}) = \bar{\theta}_i(\epsilon)\pi_i \text{ and } T = \epsilon - \tilde{\epsilon}. \tag{10}$$

I show below that this allocation is feasible and yields to strictly higher welfare than the equilibrium allocation. I have assumed that all types who get a strictly positive payoff in the complete information case will get a strictly positive payoff in the equilibrium. According to Guerrieri et al. (2010) Proposition 4, if there are positive gains from trade for all types, then all types will be active in the equilibrium, so this assumption will be automatically satisfies if there are positive gains from trade for all types.

Since $\bar{\theta}_i(\epsilon)$ and $\bar{a}_i(\epsilon)$ are all continuous in ϵ, and for $\epsilon = 0$, $\bar{U}_i(0) > 0$ for all $i \in I^*(0)$, then one can find a sufficiently small $\epsilon < \min_{i \in I^*(0)} \bar{U}_i(0)$ (also smaller than the threshold derived in the following lemma) such that all types (either $i \in I^*(0)$ or not) still get a strictly positive payoff. I fix $\epsilon > 0$ at such level. I argue that allocation $\{Y^{P\prime}, \lambda', \theta', \Gamma', T'\}$ is feasible and yields higher welfare than equilibrium.

Consider buyers' zero profit condition. Buyers in any submarket y_i' pays $\bar{p}_i(\epsilon)$ anyway. By Lemma 1 (described below), the first constraint in problem $P_i(\epsilon)$ is binding. Therefore, the buyers get exactly zero payoff. To show that sellers' maximization condition is satisfied, first note that according to Lemma 1, sellers' maximization condition is satisfied at any solution to problem $P(\epsilon)$. Under allocation $\{Y^{P\prime}, \lambda', \theta', \Gamma', T'\}$, all types in $I^*(0)$ produce the same level of production and get matched with the same market tightness as in allocation $\{Y^P, \lambda, \theta, \Gamma, T\}$. The only difference is that the payoff of all types is now $\tilde{\epsilon}$ less than that under allocation $\{Y^P, \lambda, \theta, \Gamma, T\}$. Since this amount is the same for all types, and all types get a payoff more than their outside option, $\bar{U}_i(\epsilon) \geq 0$, therefore sellers' maximization condition is also satisfied.

Now I check that the planner's budget constraint is satisfied:

$$\int_{Y^{P\prime}} q(\bar{\theta}_i(\epsilon))t'_m d\lambda'(\{y\}) - T'(1 - \int_{Y^{P\prime}} \frac{1}{\bar{\theta}_i(\epsilon)} d\lambda'(\{y\}))$$

$$= -\sum_{1}^{\tilde{i}-1} q(\bar{\theta}_i(0))\lambda(\{y'_i\})\frac{(\epsilon-\tilde{\epsilon})}{m(\bar{\theta}_i(0))} + \sum_{i\in\{i|\tilde{i}\leq i\leq I, i\in I^*\}} q(\bar{\theta}_i(\epsilon))\lambda(\{y'_i\})\frac{\tilde{\epsilon}}{m(\bar{\theta}_i(\epsilon))} - (\epsilon-\tilde{\epsilon}) \sum_{i\in\{i|\tilde{i}\leq i\leq I, i\notin I^*\}} \pi_i$$

$$= -\sum_{1}^{\tilde{i}-1} \pi_i(\epsilon-\tilde{\epsilon}) + \sum_{i\in\{i|\tilde{i}\leq i\leq I, i\in I^*\}} \pi_i\tilde{\epsilon} - (\epsilon-\tilde{\epsilon}) \sum_{i\in\{i|\tilde{i}\leq i\leq I, i\notin I^*\}} \pi_i = 0.$$

The first equality follows the fact that there are only I submarkets, so we can change the integral to sum. It also follows from construction of y'_i in equation 9. The second equality comes from the construction of λ'. The last equality follows from definition of $\tilde{\epsilon}$.

Since allocation $\{Y^{P\prime}, \lambda', \theta', \Gamma', T'\}$ is derived from $\{Y^P, \lambda, \theta, \Gamma, T\}$ by just redistribution of transfers across types in a lump-sum way, the welfare level for both allocations are the same. This will conclude the proof, because this will imply that the welfare from allocation $\{Y^{P\prime}, \lambda', \theta', \Gamma', T'\}$ is strictly higher than that in equilibrium. I prove this formally below. Let $W(\{Y^P, \lambda, \theta, \Gamma, T\})$ denote the level of welfare from the allocation $\{Y^P, \lambda, \theta, \Gamma, T\}$. Then we have:

$$W(\{Y^{P\prime}, \lambda', \theta', \Gamma', T'\} = \sum_{i=1}^{i=I} \pi_i\bar{U}'_i = W(\{Y^P, \lambda, \theta, \Gamma, T\}) - \tilde{\epsilon} = \sum_{i=1}^{i=I} \pi_i\bar{U}_i(\epsilon) - \tilde{\epsilon}$$

$$> \sum_{i\in\{i|1\leq i<\tilde{i} \text{ or } i\notin I^*\}} \pi_i(\bar{U}_i(0)+\epsilon) + \sum_{i\in\{i|\tilde{i}\leq i\leq I, i\in I^*\}} \pi_i\bar{U}_i(0) - \tilde{\epsilon}$$

$$= \sum_{i=1}^{i=I} \pi_i\bar{U}_i(0)$$

The first inequality directly follows equation 8. The next equality follows definition of $\tilde{\epsilon}$. This completes the proof.

□

I prove the following lemma, stating that at the solution to problem $P(\epsilon)$ for any ϵ, the first constraint in $P_{\tilde{i}}(\epsilon)$ should be binding. Also, I show that sellers are not attracted to submarkets designed for higher types. This lemma is similar to Lemma 1 in Guerrieri et al. (2010), but I prove a stronger claim. I prove that higher types are *strictly* worse off if they apply to submarkets designed for lower types. The reason that I get a stronger claim is that I assume strict monotonicity for $v_i(a)$ in i for every $a \in \bar{Y}$, while they just assume weak monotonicity.

Lemma 1. *There exist I^*, $\{\bar{U}_i(\epsilon)\}_{i\in\mathbb{I}}$ and $\{(\bar{\theta}_i(\epsilon), \bar{a}_i(\epsilon), \bar{t}_i(\epsilon))\}_{i\in\mathbb{I}}$ that solve problem $P(\epsilon)$. Also, there exists an $\bar{\epsilon} > 0$ such that for every $\epsilon \in [0, \bar{\epsilon})$, the following equations hod at any solution for problem $P_i(\epsilon)$ for $i \in I^*(\epsilon)$:*

$$q(\bar{\theta}_i(\epsilon))(u_i(\bar{a}_i(\epsilon)) - \bar{p}_i(\epsilon)) = k,$$

and

$$m(\bar{\theta}_i(\epsilon))(u_j(\bar{a}_i(\epsilon)) + \bar{p}_i(\epsilon)) + \delta_i \leq \bar{U}_j(\epsilon) \text{ for all } j,$$

where $\delta_i = \epsilon$ if $i < \tilde{i}$ and $\delta_i = 0$ otherwise.

Proof. **Part 1**: Existence of a solution

First fix ϵ and set $I^*(\epsilon) = \emptyset$. For $i = 1$, the objective function is continuous and the constraint set is compact. If the constraint is empty, set $\bar{U}_1(\epsilon) = \epsilon$. Otherwise, since the objective function is continuous and the constraint set is compact, $P_1(\epsilon)$ has a solution and a unique maximum. If the value of the maximum is less than ϵ, again set $\bar{U}_1(\epsilon) = \epsilon$. Otherwise, denote by $(\bar{\theta}_1(\epsilon), \bar{a}_1(\epsilon), \bar{p}_1(\epsilon))$ one of the maximizers and add 1 to the set $I^*(\epsilon)$.

I proceed by induction. By induction hypothesis, I have found $(\bar{U}_1(\epsilon), \bar{U}_2(\epsilon), ..., \bar{U}_{i-1}(\epsilon))$ and also $(\bar{\theta}_i(\epsilon), \bar{a}_i(\epsilon), \bar{p}_i(\epsilon))$ for all $i \in I^*(\epsilon)$. Again, if the constraint is empty, set $\bar{U}_i(\epsilon) = \epsilon$. Otherwise, $\bar{U}_i(\epsilon)$ is well-defined and unique. If the value of the maximum is less than ϵ, again set $\bar{U}_1(\epsilon) = \epsilon$. Otherwise, denote by $(\bar{\theta}_i(\epsilon), \bar{a}_i(\epsilon), \bar{p}_i(\epsilon))$ one of the maximizers and add i to the set $I^*(\epsilon)$.

Part 2: The first constraint in $P_k(\epsilon)$ is binding

Assume by way of contradiction that the constraint is not binding for some $i \in I^*(\epsilon)$. First note that $\bar{\theta}_i(\epsilon) > 0$ because $\bar{U}_j(\epsilon) > \delta_j$ for all $j \in I^*(\epsilon)$. According to sorting assumption, for every $\tau > 0$, there exists an $a' \in B_\tau(\bar{a}_i(\epsilon))$ such that

$$u_i(a') > u_i(\bar{a}_i(\epsilon)) \tag{11}$$

$$\text{and } u_j(a') < u_j(\bar{a}_i(\epsilon)) \text{ for all } j < i. \tag{12}$$

Set $\tau > 0$ sufficiently small such that $q(\bar{\theta}_i(\epsilon))(v_i(a') - \bar{p}_i(\epsilon)) \geq k$ for all $B_\tau(\bar{a}_i(\epsilon))$. Now consider $P_i(\epsilon)$. The first constraint is satisfied from choice of τ and other constraints are satisfied because for all $j < i$: $m(\bar{\theta}_i(\epsilon))(u_j(a') + \bar{p}_i(\epsilon)) + \delta_i < m(\bar{\theta}_i(\epsilon))(u_j(\bar{a}_i(\epsilon)) + \bar{p}_i(\epsilon)) + \delta_i \leq \bar{U}_j(\epsilon)$. But the value of the objective function is now higher: $m(\bar{\theta}_i(\epsilon))(u_i(a') + \bar{p}_i(\epsilon)) + \delta_i > m(\bar{\theta}_i(\epsilon))(u_i(\bar{a}_i(\epsilon)) + \bar{p}_i(\epsilon)) + \delta_i$, which is a contradiction with $(\bar{\theta}_i(\epsilon), \bar{a}_i(\epsilon), \bar{p}_i(\epsilon))$ being a solution to problem $P_i(\epsilon)$.

Part 3: Incentive compatibility for all types when $\epsilon = 0$

Fix an i such that $i \in I^*(\epsilon)$. In this part, I show that incentive compatibility holds at $\epsilon = 0$, that is,

$$m(\bar{\theta}_i(0))(u_j(\bar{a}_i(0)) + \bar{p}_i(0)) \leq \bar{U}_j(0) \text{ for all for all } i \in I^*(0).$$

Assume by way of contradiction that there exists a k such that $i < k$ and $m(\bar{\theta}_i(0))(u_k(\bar{a}_i(0)) + \bar{p}_i(0)) \geq \bar{U}_k(0)$. Denote the smallest such k by h. That is,

$$m(\bar{\theta}_i(0))(u_j(\bar{a}_i(0)) + \bar{p}_i(0)) < \bar{U}_j(0) \text{ for all } i \leq j < h, \tag{13}$$

$$m(\bar{\theta}_i(0))(u_h(\bar{a}_i(0)) + \bar{p}_i(0)) \geq \bar{U}_h(0). \tag{14}$$

Now I show that $(\bar{\theta}_i(0), \bar{a}_i(0), \bar{p}_i(0))$ is feasible for problem $P_h(0)$. The first constraint in problem $P_h(0)$ is satisfied because $q(\bar{\theta}_i(0))(v_h(\bar{a}_i(0)) - \bar{p}_i(0)) > q(\bar{\theta}_i(0))(v_i(\bar{a}_i(0)) - \bar{p}_i(0)) \geq k$, where the first inequality follows from strict monotonicity assumption. Also, $m(\bar{\theta}_i(0))(u_j(\bar{a}_i(0)) + \bar{p}_i(0)) \leq \bar{U}_j(0)$ holds true for any j with $i < j < k$ according to equation 13, and holds true for any j with $j \leq i$, because $(\bar{\theta}_i(0), \bar{a}_i(0), \bar{p}_i(0))$ is feasible for problem $P_i(0)$.

According to sorting assumption, there exists an b, where $(b, .) \in \bar{Y}$, sufficiently close to a_i such that $q(\bar{\theta}_i(0))(v_h(b) - \bar{p}_i(0)) \geq k$, and

$$u_h(b) > u_h(\bar{a}_i(0)), \tag{15}$$

$$\text{and } u_j(b) < u_j(\bar{\theta}_i(0)) \text{ for all } j < h. \tag{16}$$

Now, the claim is that $(\bar{\theta}_i(0), b, \bar{p}_i(0))$ is feasible for problem P_h but delivers strictly higher utility for type h. First, The first constraint is satisfied by choice of b. Second, all incentive compatibility constraints are satisfied because $m(\bar{\theta}_i(0))(u_j(b) + \bar{p}_i(0)) < m(\bar{\theta}_i(0))(u_j(\bar{a}_i(0)) + \bar{p}_i(0)) \leq \bar{U}_j(0)$ for all $j < i$, where the weak inequality follows from the fact that $(\bar{\theta}_i(0), \bar{a}_i(0), \bar{p}_i(0))$ is feasible for problem P_h. The value of the objective function is greater than $\bar{U}_h(0)$ because $m(\bar{\theta}_i(0))(u_h(b) + \bar{p}_i(0)) > m(\bar{\theta}_i(0))(u_h(\bar{a}_i(0)) + \bar{p}_i(0)) \geq \bar{U}_h(0)$, which contradicts with $\bar{U}_h(0)$ being a maximizer of P_h.

Part 4: Existence of $\tilde{\epsilon}$ and incentive compatibility for all types

Define $e(\epsilon)$ as follows: $e(\epsilon) = \min_{i,k>i}\{\bar{U}_k(\epsilon) - m(\bar{\theta}_i(\epsilon))(u_k(\bar{a}_i(\epsilon)) + p_i(\epsilon)) - \delta_i\}$. In part 3, I showed that $e(0) > 0$. Since $e(\epsilon)$ is a continuous function in ϵ, there exists a $\tilde{\epsilon} > 0$

such that $e(\epsilon) > 0$ for any $\epsilon \in [0, \tilde{\epsilon})$. Therefore, all incentive compatibility constraints are satisfied. This concludes the proof. $\square$

8.2 Proof of Result 2

Proof. I will construct an allocation in which type i sellers get matched with probability $m(\theta_i)$ and produce a_i^{CI}. Under part 5 (a) or 5 (b) of Assumption 2, it can be easily shown that that U_i^{CI} is also increasing in i. Let $\tilde{i}$ denote the highest type of sellers without gains from trade. Then all types 1, 2, ..., $\tilde{i}$ are inactive, that is, they are matched with probability 0. Given this point, I assume without loss of generality that there are positive gains from trade for all types and then I construct a feasible allocation that achieves the complete information allocation for all these types. It will become clear through our construction that due to our assumptions, no type has incentive to go to the submarkets with higher types. Therefore, even if some types are inactive, they get enough transfers which discourage them from applying to any active submarket.

Consider the following allocation:

$$Y^P = \{y_i\}_{1 \le i \le I}, \theta(y_i) = \theta_i^{CI}, \gamma_i(y_i) = 1, \lambda(\{y_i\}) = \theta_i^{CI}\pi_i, T = 0, \tag{17}$$

where y_i is defined as follows:

$$y_i = (a_i^{CI}, p_i, t_{m,i}), \tag{18}$$

Moreover $p_1 = -u_1(a_1^{CI})$, and for $i \ge 2$, p_i is defined recursively as follows:

$$m(\theta_i^{CI})(p_i + u_i(a_i^{CI})) = m(\theta_{i-1}^{CI})(p_{i-1} + u_i(a_{i-1}^{CI})). \tag{19}$$

Finally, $t_{m,i}$ is defined from the following condition:

$$q(\theta_i^{CI})(v_i(a_i^{CI}) - p_i - t_{m,i}) = k.$$

I argue that this allocation is feasible. Then I construct another allocation derived from this allocation by just redistributing transfers to maximizes the planner's objective function.

Step 1:

According to Assumption 2, θ_i^{CI}, a_i^{CI} is increasing in i. a_i^{CI} is increasing in i because $u_i(a) + v_i(a)$ satisfies increasing differences and supermodularity properties. Also, because $m(\theta)(u_i(a_i^{CI}) + v_i(a_i^{CI})) - k\theta$ satisfies increasing differences property in θ and i, therefore θ_i^{CI} is increasing in i.

Step 2: From now on in this proof, I drop the superscript CI to reduce the notation. Note that in equation 19, p_i is set such that all local downward incentive compatibility constraints are binding. That is, for all $i \geq 2$ type i is indifferent between choosing y_i and y_{i-1}. Now, I want to show that sellers' maximization constraint is satisfied.

First, I show that type $i-1$ weakly prefers y_{i-1} over y_i, that is,

$$m(\theta_{i-1})(p_{i-1} + u_{i-1}(a_{i-1})) \geq m(\theta_i)(p_i + u_{i-1}(a_i)). \tag{20}$$

I begin from the right hand side and show that the inequality holds:

$$\begin{aligned}
&m(\theta_i)(p_i + u_i(a_i)) - m(\theta_i)(u_i(a_i) - u_{i-1}(a_i)) \\
&= m(\theta_{i-1})(p_{i-1} + u_{i-1}(a_{i-1})) + m(\theta_{i-1})(u_i(a_{i-1}) - u_{i-1}(a_{i-1})) - m(\theta_i)(u_i(a_i) - u_{i-1}(a_i)) \\
&\leq m(\theta_{i-1})(p_{i-1} + u_{i-1}(a_{i-1})) + m(\theta_i)(u_i(a_{i-1}) - u_{i-1}(a_{i-1}) - u_i(a_i) + u_{i-1}(a_i)) \\
&\leq m(\theta_{i-1})(p_{i-1} + u_{i-1}(a_{i-1}))
\end{aligned}$$

The first equality follows from construction of p_i (equation 19). The first inequality follows the fact that θ_i and $u_i(.)$ are both increasing in i. The second inequality follows from single crossing property of u in $(a; i)$ and also from the fact that $a_{i-1} \leq a_i$ (component by component).

Second, I explicitly calculate p_i in terms of p_1:

$$\begin{aligned}
&m(\theta_i^{CI})(p_i + u_i(a_i^{CI})) = m(\theta_{i-1}^{CI})(p_{i-1} + u_i(a_{i-1}^{CI})) \\
&= m(\theta_{i-1}^{CI})(p_{i-1} + u_{i-1}(a_{i-1}^{CI})) + m(\theta_{i-1}^{CI})(u_i(a_{i-1}^{CI})) - u_{i-1}(a_{i-1}^{CI})) \\
&= m(\theta_{i-1}^{CI})(p_{i-1} + u_{i-1}(a_{i-1}^{CI})) + K_i(\theta_{i-1}^{CI}, a_{i-1}^{CI}) - K_{i-1}(\theta_{i-1}^{CI}, a_{i-1}^{CI}),
\end{aligned} \tag{21}$$

where $K_i(\theta, a)$ is defined as follows: $K_i(\theta, a) = m(\theta)u_i(a)$.

Using telescoping technique yields the following for all $i \geq 2$:

$$m(\theta_i^{CI})(p_i + u_i(a_i^{CI})) = m(\theta_1^{CI})(p_1 + u_1(a_1^{CI})) + \sum_{j=2}^{i}[K_j(\theta_{j-1}^{CI}, a_{j-1}^{CI}) - K_{j-1}(\theta_{j-1}^{CI}, a_{j-1}^{CI})]$$

Now, I show that for all i and k with $k \leq i-1$, type k does not gain by choosing submarket y_i, that is,

$$m(\theta_k^{CI})(p_k + u_k(a_k^{CI})) \geq m(\theta_i^{CI})(p_i + u_k(a_i^{CI}).$$

Note that

$$\begin{aligned}
&m(\theta_k^{CI})(p_k + u_k(a_k^{CI})) - m(\theta_i^{CI})(p_i + u_k(a_i^{CI})) \\
&= \sum_{j=k}^{i-1} \Big[m(\theta_j^{CI})(p_j + u_j(a_j^{CI})) - m(\theta_{j+1}^{CI})(p_{j+1} + u_j(a_{j+1}^{CI}))
\end{aligned}$$

$$
\begin{aligned}
&+m(\theta_{j+1}^{CI})u_j(a_{j+1}^{CI}) - m(\theta_j^{CI})u_j(a_j^{CI}) \\
&\left.-m(\theta_{j+1}^{CI})u_k(a_{j+1}^{CI}) + m(\theta_j^{CI})u_k(a_j^{CI})\right] \\
\geq \sum_{j=k}^{i-1} &\left[m(\theta_{j+1}^{CI})u_j(a_{j+1}^{CI}) - m(\theta_j^{CI})u_j(a_j^{CI})\right. \\
&\left.-m(\theta_{j+1}^{CI})u_k(a_{j+1}^{CI}) + m(\theta_j^{CI})u_k(a_j^{CI})\right]
\end{aligned}
$$

$$
\geq \sum_{j=k}^{i-1} \left[m(\theta_j^{CI})(u_j(a_{j+1}^{CI}) - u_k(a_{j+1}^{CI}) - u_j(a_j^{CI}) + u_k(a_j^{CI}))\right] \geq 0.
$$

The first equality is derived by doing some algebra and using telescoping technique. The first inequality uses the fact that type $i-1$ weakly prefers y_{i-1} over y_i (see equation 20). The second inequality uses $\theta_{j+1}^{CI} \geq \theta_j^{CI}$ and also the fact that u_i is increasing in i for every a. The last inequality is the implication of sorting assumption on u (the first part of Assumption 1) and the fact that $a_{j+1}^{CI} \geq a_j^{CI}$.

If $i+1 \leq k$, I use the same technique as above:

$$
\begin{aligned}
&m(\theta_k^{CI})(p_k + u_k(a_k^{CI})) - m(\theta_i^{CI})(p_i + u_k(a_i^{CI})) \\
= \sum_{j=i+1}^{k} &\left[m(\theta_j^{CI})(p_j + u_j(a_j^{CI})) - m(\theta_{j-1}^{CI})(p_{j-1} + u_j(a_{j-1}^{CI}))\right. \\
&+m(\theta_{j-1}^{CI})u_j(a_{j-1}^{CI}) - m(\theta_j^{CI})u_j(a_j^{CI}) \\
&\left.-m(\theta_{j-1}^{CI})u_k(a_{j-1}^{CI}) + m(\theta_j^{CI})u_k(a_j^{CI})\right] \\
\geq \sum_{j=i+1}^{k} &\left[m(\theta_{j-1}^{CI})u_j(a_{j-1}^{CI}) - m(\theta_j^{CI})u_j(a_j^{CI})\right. \\
&\left.-m(\theta_{j-1}^{CI})u_k(a_{j-1}^{CI}) + m(\theta_j^{CI})u_k(a_j^{CI})\right]
\end{aligned}
$$

$$
\geq \sum_{j=k}^{i-1} \left[m(\theta_{j-1}^{CI})(u_j(a_{j-1}^{CI}) - u_k(a_{j-1}^{CI}) - u_j(a_j^{CI}) + u_k(a_j^{CI}))\right] \geq 0.
$$

The first equality is again derived by using telescoping technique. The first inequality follows construction of t_i. The second inequality uses $\theta_{j+1}^{CI} \geq \theta_j^{CI}$ and also the fact that u_i is increasing in i for every a. The last inequality is the implication of sorting assumption on u and the fact that $a_{j+1}^{CI} \geq a_j^{CI}$.

To show $U_i \geq 0$ for all i, consider equation 21. The first term in the right hand side is zero following the definition of t_1. The summation is positive following the assumption that u_i is increasing in i for every a.

Step 3:

From construction of $t_{m,i}$, buyers' zero profit condition is satisfied.

Step 4:

Consider the budget constraint:

$$\begin{aligned}
B &= \int_{Y^P} \Big[q(\theta(y))t_m\Big] d\lambda(\{y\}) \\
&= \sum \pi_i \Big[m(\theta_i)(v_i(a_i) - p_i) - k\theta_i\Big] \\
&= \sum_{i=1}^{I} \pi_i \Big[m(\theta_i)(v_i(a_i) + u_i(a_i)) - k\theta_i\Big] - \sum_{i=1}^{I} \pi_i \Big[m(\theta_i)(p_i + u_i(a_i))\Big] \\
&= \sum_{i=1}^{I} \pi_i \Big[m(\theta_i)(v_i(a_i)+u_i(a_i))-k\theta_i\Big] - \sum_{i=1}^{I} \pi_i \Big[m(\theta_1)(p_1+u_1(a_1))+\sum_{j=2}^{i} \big[K_j(\theta_{j-1}, a_{j-1})-K_{j-1}(\theta_{j-1}, a_{j-1})\big]\Big] \\
&= \sum_{i=1}^{I} \pi_i \Big[m(\theta_i)(v_i(a_i)+u_i(a_i))-k\theta_i\Big] - \sum_{i=1}^{I} \pi_i \Big[\sum_{j=2}^{i} \big[K_j(\theta_{j-1}, a_{j-1})-K_{j-1}(\theta_{j-1}, a_{j-1})\big]\Big] - m(\theta_1)(p_1+u_1(a_1))
\end{aligned} \tag{22}$$

where I used used definition of p_i for the fourth equality.

Now, suppose part $5(a)$ **of Assumption 2 holds.** I want to show that $B \geq 0$. To do that, I prove that the elements in the summation (equation 22) for each i are positive. I proceed with induction on i. If $I = 1$, then $B = 0$ trivially by the choice of p_1. For $I = 2$:

$$\begin{aligned}
m(\theta_2)(v_2(a_2) + u_2(a_2)) - k\theta_2 &\geq m(\theta_1)(v_2(a_1) + u_2(a_1)) - k\theta_1 \\
&\geq m(\theta_1)(u_2(a_1) - u_1(a_1)) + m(\theta_1)(v_2(a_1) + u_1(a_1)) - k\theta_1 \\
&\geq m(\theta_1)(u_2(a_1) - u_1(a_1)) + m(\theta_1)(v_1(a_1) + u_1(a_1)) - k\theta_1 + \\
&\geq m(\theta_1)(u_2(a_1) - u_1(a_1)) + m(\theta_1)(u_2(a_1) + p_1)
\end{aligned}$$

The first inequality is due to the fact that θ_1 and a_1 are feasible for the second type maximization problem ($\max_{\theta,a}\{m(\theta)(v_2(a) + u_2(a)) - k\theta\}$). The second inequality is because v is increasing in i. The last inequality holds due to the construction of p_1.

Now assume that the induction hypothesis for type $i - 1$ is correct. Then, I show that the hypothesis will be correct for type i as well:

$$m(\theta_i)(v_i(a_i) + u_i(a_i)) - k\theta_i \geq m(\theta_{i-1})(v_i(a_{i-1}) + u_i(a_{i-1})) - k\theta_{i-1}$$

$$= m(\theta_{i-1})(v_i(a_{i-1}) + u_{i-1}(a_{i-1})) - k\theta_{i-1} + m(\theta_{i-1}(u_i(a_{i-1}) - u_{i-1}(a_{i-1}))$$

$$\geq m(\theta_{i-1})(v_{i-1}(a_{i-1}) + u_{i-1}(a_{i-1})) - k\theta_{i-1} + m(\theta_{i-1}(u_i(a_{i-1}) - u_{i-1}(a_{i-1}))$$

$$\geq m(\theta_1)(p_1+u_1(a_1))+\sum_{j=2}^{i-1}\left[K_j(\theta_{j-1}, a_{j-1})-K_{j-1}(\theta_{j-1}, a_{j-1})\right]+K_i(\theta_{i-1}, a_{i-1})-K_{i-1}(\theta_{i-1}, a_{i-1})$$

$$= m(\theta_1)(p_1 + u_1(a_1)) + \sum_{j=2}^{i}\left[K_j(\theta_{j-1}, a_{j-1}) - K_{j-1}(\theta_{j-1}, a_{j-1})\right]$$

Similar to the case for $I = 2$, the first inequality is because θ_{i-1} and a_{i-1} are feasible for type i maximization problem ($\max_{\theta,a}\{m(\theta)(v_i(a) + u_i(a)) - k\theta\}$). The second inequality is because v is increasing in i. The last inequality holds due to the induction hypothesis.

Now, suppose parts $5(b)$ **and** $5(c)$ **of Assumption 2 hold.** Again, I want to show that $B \geq 0$. The budget balance condition can be written in the following form (following equation 22):

$$B = \sum_{i=1}^{I}\pi_i\left[m(\theta_i)(v_i(a_i)+u_i(a_i))-k\theta_i\right]-\sum_{j=2}^{I}\left[\left[K_j(\theta_{j-1}, a_{j-1})-K_{j-1}(\theta_{j-1}, a_{j-1})\right]\sum_{i=j}^{I}\pi_i\right]-m(\theta_1)(p_1+u_1(a_1))$$

$$= \sum_{i=1}^{I}\pi_i\left[m(\theta_i)(v_i(a_i)+u_i(a_i))-k\theta_i\right]-\sum_{j=2}^{I}\left[\pi_j\left[K_j(\theta_{j-1}, a_{j-1})-K_{j-1}(\theta_{j-1}, a_{j-1})\right]\frac{\Delta_j}{\pi_j}\right]-m(\theta_1)(p_1+u_1(a_1))$$

$$= \sum_{i=2}^{I}\pi_i\left[m(\theta_i)(v_i(a_i)+u_i(a_i))-k\theta_i-\left[K_j(\theta_{j-1}, a_{j-1})-K_{j-1}(\theta_{j-1}, a_{j-1})\right]\frac{\Delta_i}{\pi_i}\right]+\pi_1 U_1^{CI}-m(\theta_1)(p_1+u_1(a_1)) \geq 0$$

For the second equality, I changed the order of integrals for the double sigma term. For the third equality, I defined a new variable Δ_i as follows

$$\Delta_i = \sum_{k=i}^{I}\pi_k$$

to simplify the expression. I used part 3 of Assumption 2 and choice of p_1 to establish the inequality in the last line.

Step 5:

Finally, for all i, let $p_i' = p_i + \frac{B}{m(\theta_i^{CI})}$ and $t_{m,i}' = t_{m,i} - \frac{B}{m(\theta_i^{CI})}$. Now consider the following allocation:

$$Y^P = \left\{y_i'\right\}_{1\leq i\leq I}, \theta(y_i) = \theta_i^{CI}, \gamma_i(y_i) = 1, \lambda(\{y_i\}) = \theta_i^{CI}\pi_i. \tag{23}$$

where y_i' is defined as follows:

$$y_i' = (a_i^{CI}, p_i', t_{m,i}', 0). \tag{24}$$

This allocation is feasible due to the following reasons: It's easy to check that now budget constraint is satisfied with equality (similar to the proof of the first lemma). Since $p_i' + t_{m,i}' = p_i + t_{m,i}$ for all i, buyers' zero profit is also satisfied. Sellers' maximization is still satisfied, because in the new allocation, all sellers get an expected increase of B equally over all submarkets, so their maximization decisions are not affected.

This allocation is constrained efficient because the level of θ and a assigned to every type is exactly equal to that under complete information allocation, so it is not possible to increase the value of the objective function any more. The proof is complete. □

8.3 Proofs of Asset Market with Lemons

I first prove the following lemma stating that the planner's budget constraint must be binding. This lemma is general, but I particularly use it in the proof of the asset market with lemons.

Lemma 2. *In any constrained efficient allocation, the budget constraint must hold with equality.*

Proof. I assume in this proof that all types are active, that is, all types apply to some submarket. Therefore, T does not show up in the proof. The case can be proved in a similar way otherwise. Suppose budget constraint holds but is not binding, that is, $B \equiv \int_{Y^P} \left[q(\theta(y))t_m\right] d\lambda(\{y\}) > 0$. I propose another allocation which increases the value of the objective function and keep all other constraints satisfied: For any $y \in Y^P$, consider a new y' with the same a, but with a new t_s' and t_m':

$$t_s' = t_s + \frac{B}{m(\theta(y))}\left(\int_{Y^P} \frac{d\lambda(\{y\})}{\theta(y)}\right)^{-1},$$

$$t_m' = t_m - \frac{B}{m(\theta(y))}\left(\int_{Y^P} \frac{d\lambda(\{y\})}{\theta(y)}\right)^{-1}.$$

All other elements of this new allocation is the same: for any y', $\Gamma(y') = \Gamma(y)$ and $\theta(y') = \theta(y)$ and $\lambda(y') = \lambda(y)$. That is, the new allocation is just different from the old allocation in transfers (t_s and t_m) in submarkets. Now I want to check that this allocation is also feasible.

Take any seller. The expected amount that the seller gets has increased by fixed amount of $B(\int_{Y^P} \frac{d\lambda(\{y\})}{\theta(\tilde{y})})^{-1}$ over all submarkets, so the seller does not have any incentive to change the submarkets he used to join under the old allocation. Therefore, seller's maximization condition is satisfied.

Regarding buyer's zero profit, note that that the expected amount that buyer pays remain the same because $t'_s + t'_m = t_s + t_m$.

Now I calculate planner's budget under new allocation:

$$\int_{Y'^P} \big[q(\theta(y'))t'_m\big] d\lambda(\{y'\})$$

$$= \int_{Y'^P} \big[q(\theta(y'))(t_m - \frac{B}{m(\theta(y'))}(\int_{Y^P} \frac{d\lambda(\{\tilde{y}\})}{\theta(\tilde{y})})^{-1})\big] d\lambda(\{y'\})$$

$$= \int_{Y'^P} \big[q(\theta(y'))t_m\big] d\lambda(\{y'\}) - B(\int_{Y'^P} \frac{d\lambda(\{\tilde{y}\})}{\theta(\tilde{y})})^{-1} \int_{Y'^P} \frac{d\lambda(\{y\})}{\theta(y)} = B - B = 0.$$

Therefore, the budget constraint holds. Note that the value of the objective function has increased exactly by amount B, because the planner have distributed all resources across sellers. Formally, the value of the objective function has increased by:

$$\int_{Y'^P} q(\theta(y'))(t_m - t'_m) d\lambda(\{y'\}$$

$$= \int_{Y'^P} \big[q(\theta(y'))\frac{B}{m(\theta(y'))}(\int_{Y^P} \frac{d\lambda(\{\tilde{y}\})}{\theta(\tilde{y})})^{-1}\big] d\lambda(\{y'\}) = B.$$

This concludes the proof. □

Corollary 1. *For any feasible allocation where budget constraint is not binding, that is,* $B \equiv \int_{Y^P} \big[q(\theta(y))t_m\big] d\lambda(\{y\}) > 0$, *there exists another feasible allocation* $\{\lambda', Y'^P, \theta', \Gamma'\}$ *which increases the welfare by exactly amount* B *and the budget constrains is now binding. New allocation is given by:*

$$Y'^P = \{y' | y' = (a, t_s + \frac{\Delta}{m(\theta(y))}, t_m - \frac{\Delta}{m(\theta(y))}), (a, t_s, t_m) \in Y^P\},$$

where $\Delta \equiv B(\int_{Y^P} \frac{d\lambda(\{y\})}{\theta(y)})^{-1}$.

Also, $\lambda'(\{y'\}) = \lambda(\{y\})$, $\theta'(\{y'\}) = \theta(\{y\})$ *and* $\Gamma'(\{y'\}) = \lambda(\{y\})$ *where* $y' = (a, t_s + \frac{\Delta}{m(\theta(y))}, t_m - \frac{\Delta}{m(\theta(y))})$ *for some* $y \equiv (a, t_s, t_m) \in Y^P$.

Proof of Asset market with lemons if $\pi_1 b_1 + \pi_2 b_2 \geq c_2$. Here the complete information allocation is achievable through a pooling allocation. Seller's maximization problem is satisfied, because there is just one submarket. Also, both types get positive payoff. Buyers' zero profit condition is satisfied by construction of $t_{m,i}$. Budget balance is also trivially satisfied because the planner does not make any transfers. The objective function is maximized because the θ and α allocated to both types is the same as what they get under complete information. The proof is complete. □

Proof of Asset market with lemons when $\pi_1 b_1 + \pi_2 b_2 < c_2$. Here the complete information allocation is not achievable through a pooling allocation, because type two gets strictly negative payoff, therefore, pooling allocation is not feasible. If $b_2 - c_2$ is greater than $b_1 - c_1$, part 5(b) of Assumption 2 is violated. If $b_2 - c_2$ is less than or equal to $b_1 - c_1$, then it's easy to check that although part 5(b) is satisfied, part 5 is violated. Therefore, I cannot use result 2. Hence, I need to solve the planner's problem completely, taking all constraints into account. I proceed in 6 steps. In the first step, I simplify the problem. Specifically, I show that we can assume without loss of generality that there are only two submarkets. In the second step, I show that the market tightness in both submarkets must be strictly positive. In the third step, I show that the market tightness in both submarkets must be less than or equal to 1. In the fourth step, I show that α (probability that the seller gives the asset to the buyer) in both submarkets must be equal to 1. In the fifth step, I show that market tightness in the submarket that type one applies to must be equal to 1. In the last step, I calculate the market tightness in the other submarket and conclude the characterization of the constrained efficient allocation.

Step 1: Simplifying the problem

According to Lemma 2, the planner's budget constraint must be binding. Moreover, we can assume without loss of generality that different types are allocated to different submarkets. Let $y_i \equiv (\alpha_i, p_i, t_{m,i})$ denote the submarket that type i applies to. Also, let $\theta_i \equiv \theta(y_i)$. Now, I can write planner's problem as follows:

Problem 3 (Asset market with lemons, 1)**.**

$$\max_{\{(\theta_i,\alpha_i,p_i,t_{m,i})\}_{i=1,2}} \sum_{1}^{2} \pi_i(\min\{\theta_i, 1\}\alpha_i(h_i - c_i) - k\theta_i),$$

subject to

$$\min\{\theta_i^{-1}, 1\}(\alpha_i h_i - p_i - t_{m,i}) = k, \ \textit{for all } i$$

$$\min\{\theta_1, 1\}(p_1 - \alpha_1 c_1) \geq \min\{\theta_2, 1\}(p_2 - \alpha_2 c_1) \ \textit{(IC-12)}\ ,$$

$$\min\{\theta_2, 1\}(p_2 - \alpha_2 c_2) \geq \min\{\theta_1, 1\}(p_1 - \alpha_1 c_2) \ \textit{(IC-21)}\ ,$$

$$\min\{\theta_1, 1\}(p_1 - \alpha_1 c_1) \geq 0 \ \textit{(IR-1)}$$

$$\min\{\theta_2, 1\}(p_2 - \alpha_2 c_2) \geq 0 \ \textit{(IR-2) and}$$

$$\sum_{1}^{2} \pi_i \min\{\theta_i, 1\} t_{m,i} = 0 \ \textit{(BB).}$$

To write the objective function, I consider the fact that the planner's budget constraint is binding. Therefore, no transfers appear in the objective function. The first constraint is buyers' zero profit condition. I break sellers' maximization condition into two parts. The first part of sellers' maximization condition includes sellers' Incentive compatibility constraints: In the second (the third) line, I ensure that type one (2) does not want to apply to the submarket designed for the other type. I call this constraint IC-12 (IC-21). The second part of sellers' maximization condition includes basically sellers' participation or individual rationality constraints: In the fourth (fifth) line, I ensure that type one (2) gets strictly positive payoff. I call this constraint IR-1 (IR-2). The last line is planner's budget constraint.

Step 2: $\theta_1 > 0$ and $\theta_2 > 0$

If both θ_1 and θ_2 are 0 then the welfare is 0. But this is not possible because we know at least that equilibrium allocation is feasible and delivers strictly positive utility. To rule out the case that even one of them cannot be 0, first note that IC-12 and IC-21 together imply that:

$$(m(\theta_1)\alpha_1 - m(\theta_2)\alpha_2)c_1 \leq m(\theta_1)p_1 - m(\theta_2)p_2 \leq (m(\theta_1)\alpha_1 - m(\theta_2)\alpha_2)c_2. \tag{25}$$

But $c_1 < c_2$, therefore,

$$m(\theta_1)\alpha_1 \geq m(\theta_2)\alpha_2. \tag{26}$$

If $\theta_1 = 0$, then θ_2 must be 0 as well, which leads to 0 welfare. Nevertheless, this cannot be part of a planner's allocation, given the fact that the equilibrium allocation is feasible and delivers strictly positive welfare. Thus $\theta_1 > 0$. If $\theta_2 = 0$, then it's easy to check that the maximum possible welfare in this case (even if $\theta_1 = 1$) is less than the welfare under the proposed planner's allocation.

Let $r_i \equiv \min\{\theta_i, 1\}p_i$ for all i. For any θ_i and $r_i \in \mathbb{R}$, we can find a unique $p_i \in \mathbb{R}$ which solves the maximization problem. From now on, we work with r_i instead because it simplifies the analysis. Also, we solve buyers' zero profit condition for $t_{m,i}$ in terms of other variables to remove $t_{m,i}$ from the problem completely and then rewrite the problem as follows:

Problem 4 (Asset market with lemons, 2)**.**

$$\max_{\{(\theta_i,\alpha_i,r_i)\}_{i=1,2}} \sum_{1}^{2} \pi_i(\min\{\theta_i, 1\}\alpha_i(h_i - c_i) - k\theta_i),$$

subject to

$$r_1 - \min\{\theta_1, 1\}\alpha_1 c_1 \geq r_2 - \min\{\theta_2, 1\}\alpha_2 c_1 \text{ (IC-12)},$$

$$r_2 - \min\{\theta_2, 1\}\alpha_2 c_2 \geq r_1 - \min\{\theta_1, 1\}\alpha_1 c_2 \text{ (IC-21)},$$

$$r_1 - \min\{\theta_1, 1\}\alpha_1 c_1 \geq 0 \textit{ (IR-1)}$$

$$r_2 - \min\{\theta_2, 1\}\alpha_2 c_2 \geq 0 \textit{ (IR-2) and}$$

$$\sum_1^2 \pi_i(\min\{\theta_i, 1\}\alpha_i h_i - k\theta_i - r_i) = 0 \textit{ (BB).}$$

Step 3: $\theta_1 \leq 1$ **and** $\theta_2 \leq 1$

Suppose $\theta_i > 1$ for some i, now I propose the following: $\theta_i' = 1$, $r_i' = r_i + k(\theta_i - 1)\pi_i$ and $r_j' = r_j + k(\theta_i - 1)\pi_i$ where $j \neq i$. Therefore, if I replace θ_i, r_1 and r_2 by $\theta_i' = 1$, r_1' and r_2', I can increase the value of the objective function by $k(\theta_i - 1)$. Also, the new solution satisfies all the constraints because of the following: Obviously, IC-12 and IC-21 are still satisfied, because the change in r_1 is the same as the change in r_2 and also $\min\{\theta_1, 1\}$ and $\min\{\theta_2, 1\}$ have not changed. IR-1 and IR-2 are satisfied because $r_1' > r_1$ and $r_2' > r_2$. BB is also satisfied by construction of r_1' and r_2'. A contradiction. Therefore, for all $i \in \{1, 2\}$, $\theta_i \leq 1$.

Step 4: $\alpha_1 = \alpha_2 = 1$

Suppose $\alpha_i < 1$ for some i. Let α_i' be defined such that $\alpha'(\theta_i - \epsilon)$ equals $\alpha\theta_i$, where $0 < \epsilon < \theta_i(1 - \alpha_i)$. Now consider the following: $\theta_i' = \theta_i - \epsilon$, $r_i' = r_i + k\epsilon\pi_i$ and $r_j' = r_j + k\epsilon\pi_i$ where $j \neq i$.

Now, if I replace α_i, θ_i, r_1 and r_2 by α_i', θ_i', r_1' and r_2', I can increase the value of the objective function by $k\epsilon$. But it's a contradiction if I show that the new solution satisfies all the constraints. But the new solution indeed satisfies all the constraints because of the following: Obviously, IC-12 and IC-21 are still satisfied, because $\min\{\theta_i, 1\}\alpha_i = \min\{\theta_i', 1\}\alpha_i'$. IR-1 and IR-2 are satisfied because $r_1' > r_1$ and $r_2' > r_2$. BB is also satisfied by construction of r_1' and r_2'. A contradiction. Therefore, for all $i \in \{1, 2\}$, $\alpha_i = 1$.

Step 5: $\theta_1 = 1$

For simplicity, I write the planner's problem again incorporating the results so far:

Problem 5 (Asset market with lemons, 2)**.**

$$\max_{\{(\theta_i, r_i)\}_{i=1,2}} \sum_1^2 \pi_i(\theta_i(h_i - c_i) - k\theta_i),$$

subject to

$$r_1 - \theta_1 c_1 \geq r_2 - \theta_2 c_1 \textit{ (IC-12)},$$

$$r_2 - \theta_2 c_2 \geq r_1 - \theta_1 c_2 \textit{ (IC-21)},$$

$$r_1 - \theta_1 c_1 \geq 0 \textit{ (IR-1)}$$

$$r_2 - \theta_2 c_2 \geq 0 \textit{ (IR-2) and}$$

$$\sum_{1}^{2} \pi_i(\theta_i h_i - k\theta_i - r_i) = 0 \text{ (BB)}.$$

First note that $\theta_1 \geq \theta_2$ following equation 26 and because $\alpha_1 = \alpha_2 = 1$ according to step 4. By way of contradiction, assume that $\theta_1 < 1$ at a solution. I consider two cases. First, assume that IR-2 is not binding. I propose the following: $\theta'_i = \theta_i + \epsilon$ for all i where $\epsilon < \min\{1 - \theta_1, \frac{r_2 - \theta_2 c_2}{c_2 - \pi_1 b_1 - \pi_2 b_2}\}$, and $r'_i = r_i + (\pi_1 b_1 + \pi_2 b_2)\epsilon$ for all i. It's easy to check that all constraints are satisfied, but the value of the objective function now has increased by $(\pi_1(b_1 - c_1) + \pi_2(b_2 - c_2))\epsilon$. A contradiction. Note that I used equation 26 to ensure that $\theta_2 + \epsilon < 1$.

Therefore, IR-2 is binding. I propose the following: $\theta'_1 = \theta_1 + \epsilon$ where $\epsilon < 1 - \theta_1$, $r'_1 = r_1 + b_1\epsilon$. It's again easy to check that all constraints are satisfied. The only tricky thing here is to check that IC-21 is satisfied. But The LHS in IC-21 is fixed. The RHS increases by $\epsilon(b_1 - c_2)$ which is a negative number, so IC-21 is not violated. (Note that $b_1 - c_2 < 0$, otherwise $\pi_1 b_1 + \pi_2 b_2 > \pi_1 c_2 + \pi_2 c_2 = c_2$ which contradicts the initial assumption that $\pi_1 b_1 + \pi_2 b_2 < c_2$). But the the value of the objective function now has increased by $\pi_1 b_1 \epsilon$. A contradiction.

Step 6: Calculating θ_2 and the rest of unknowns

I write r_1 from the budget constraint in terms of other variables, specially r_2:

$$r_1 = b_1 + \frac{\pi_2}{\pi_1}\theta_2 b_2 - \frac{\pi_2}{\pi_1} r_2 \tag{27}$$

Now, one can write equation 25 as follows after replacing r_1 from the above equation:

$$(1 - \theta_2)c_1 \leq b_1 + \frac{\pi_2}{\pi_1}\theta_2 b_2 - \frac{r_2}{\pi_1} \leq (1 - \theta_2)c_2. \tag{28}$$

First, note that IR-1 is implied by IC-12 and IR-2. Second, I argue that IR-2 must be binding at the solution. By way of contradiction, suppose not. Then only equation 28 is sufficient to determine θ_2. But in order to maximize the objective function, I need to choose the highest possible θ_2 consistent with equation 28, which is $\theta_2 = 1$. But according to equation 28, $r_2 = \pi_1 b_1 + \pi_2 b_2$ and $r_2 > c_2$ from IR-2, which is a contradiction with $\pi_1 b_1 + \pi_2 b_2 < c_2$. Therefore, IR-2 is binding. Third, since IR-2 is binding, I replace r_2 by $\theta_2 c_2$ and rewrite equation 28 again:

$$(1 - \theta_2)c_1 \leq b_1 + \frac{\theta_2}{\pi_1}(\pi_2 b_2 - c_2) \leq (1 - \theta_2)c_2. \tag{29}$$

Now, it's easy to see that the right inequality in 29 is satisfied for any $\theta_2 \in [0, 1]$, because $b_1 < c_2$. In order to maximize the objective function, I need to find the maximum value for

θ_2 under which the left inequality in 29 is satisfied ($(1-\theta_2)c_1 \leq b_1 + \frac{\theta_2}{\pi_1}(\pi_2 b_2 - c_2)$). This implies that $\theta_2 = \frac{\pi_1(b_1-c_1)}{c_2-\pi_2 b_2-\pi_1 c_1}$.

Now the proof is complete, because I have found the values for α_i, θ_i and r_i. I can calculate values of p_i from θ_i and r_i. I can calculate values $t_{m,i}$ from buyers' zero profit conditions and check that they are the same as them in Table 1.

$\square$

What if there are no gains from trade for some types? Because the proof is similar to the the previous proof, I do not repeat it here. I just give the sketch of the proof. Consider the equilibrium characterization. Suppose the planner compensates type one by some small amount $\epsilon > 0$. This will relax the constraint of maximization problem for type two and create some resources. The question is that whether these resources are enough to cover the transfers to type one so that the budget constraint of the problem is satisfied. The following is the maximization problem for type two :

$$\max_{\theta,\alpha} m(\theta)\alpha(h_2 - c_2) - \theta k,$$

subject to

$$m(\theta)\alpha(h_2 - c_1) - \theta k = \epsilon.$$

The constraint has been obtained from solving p from the free entry condition and then writing the IC constraint without p.

Now, I solve α from the constraint and write the objective function again. After simplification, the problem turns into the following:

$$\max_{\theta} \frac{h_2 - c_2}{h_2 - c_1}\epsilon - \frac{c_2 - c_1}{h_2 - c_1}k\theta,$$

subject to

$$0 \leq \frac{\epsilon + \theta k}{m(\theta)(h_2 - c_1)} \leq 1$$

where the constraint is due to the fact that α is a probability. Since the objective function is decreasing in θ, then the solution is $\theta = \frac{\epsilon}{h_2-c_1-k}$ and $U_2 = \frac{\epsilon(b_2-c_2)}{b_2-c_1}$. The planner needs to raise at least $\pi_1\epsilon$ to cover the subsidies to type one from type two sellers. That is, $\pi_2 U_2 \geq \pi_1\epsilon$. This is equivalent to $\pi_2 \geq \frac{2b_2-c_1-c_2}{b_2-c_1}$. In other words, if the share of type two in the population is small, the market will shut down and the planner cannot save the market. I show if there are no gains from trade for one type and if this is contagious, that is, if this leads to inactivity of another type in the equilibrium, then the planner also might not be able to increase the welfare. $\square$

8.4 Proof of the Rat Race

Proof. This proposition is basically a special case of result 2. It is straight forward to check that all conditions are satisfied. Specially note that, part 5(a) of Assumption 2 is satisfied, therefore, we do not need any assumption on the distribution. □

8.5 Asset Market with Continuous Type Space

8.5.1 Preliminaries of the Proofs

First of all, note that the ideas used here are the same those used in the simple two-type model (asset market with lemons). However, mathematical tools that I use here are different, because the state space is continuous.

One way of proving Proposition 5 is to take a Guess-And-Verify approach. I guess that the complete information allocation is achievable. Then I check whether conditions for feasibility are satisfied. One problem is that if complete information allocation is not achievable (like Proposition 6), this approach does not work, because checking for feasibility is not sufficient, since there might be other feasible allocations which deliver a higher value of the objective function for the planner. Therefore, in order to be able to use a general solution method, I first characterize incentive compatible schemes, as is common in the mechanism design literature. Then, I work with a modified problem in which sellers' maximization condition has been replaced by some other constraints (monotonicity and Envelope condition). See below for the details.

I proceed in 6 steps. In the first step, I use the first and second lemma to simplify the problem. Specifically, I show that we can assume without loss of generality that there are only two submarkets. In the second step, I show that the market tightness in both submarkets must strictly positive. In the third step, I show that the market tightness in both submarkets must be less than or equal to 1. In the fourth step, I show that α (probability that the seller gives the asset to the buyer) in both submarkets must be equal to 1. In the fifth step, I show that market tightness in the submarket that type one applies to must be equal to 1. In the last step, I calculate the market tightness in the other submarket and introduce the constrained efficient allocation.

Step 1: Simplifying the problem

The planner's budget constraint must be binding. Suppose for the constrained efficient allocation, the budget constraint is not binding, that is,

$$B \equiv \int_{\text{supp } G} \left[q(\theta_{GH}(\phi))t_m(\phi)\right] dG > 0.$$

For any ϕ, define new t'_s and t'_m as follows:

$$t'_s(\phi) = t_s(\phi) + \frac{B}{m(\theta(\phi))}(\int_{\text{supp}G} \frac{dG}{\theta(\phi)})^{-1},$$

$$t'_m(\phi) = t_m(\phi) - \frac{B}{m(\theta(\phi))}(\int_{\text{supp}G} \frac{dG}{\theta(\phi)})^{-1}.$$

Both G, H remain the same, so the new allocation is just different from the old original one in transfers (t_s and t_m). In this new allocation, the maximization problem of sellers and zero profit of buyers do not change. It's similar to the discrete type case that the planner's budget constraint now holds with equality, but the value of the objective function has increased by amount B. A contradiction with the original allocation to be constrained efficient.

Also, similar to discrete type case, we can assume without loss of generality that different types are allocated to different submarkets. To show that ignoring pooling allocations is without loss of generality, suppose just two types are allocated to one submarket ϕ with strictly positive probability. Then, it is easy to construct another allocation and allocate those two types to two different submarkets and adjust t_m for those submarkets accordingly. The market tightness and level of production for both submarkets remain the same.

So far I have shown that any constrained efficient allocation can be assumed to be separating without loss of generality. Now I want to show that any allocation can be implemented without any lotteries. Assume by way of contradiction that the constrained efficient allocation involves lotteries, that is, there exists some type z which is assigned to ϕ_1 with probability $\alpha > 0$ and to ϕ_2 with probability $\beta > 0$. This seller gets $\alpha m(\theta_1)(p_1 - c(y)) + \beta m(\theta_2)(p_2 - c(y))$. Replace these two submarkets with ϕ_3 with market tightness $m(\theta_3) = \alpha m(\theta_1) + \beta m(\theta_2)$ and transfers $t_{s,3} = \frac{\alpha m(\theta_1)p_1 + \beta m(\theta_2)p_2}{(\alpha+\beta)m(\theta_3)}$ and $t_{m,3} = h(z) - t_{s,3} - \frac{k}{3}$. The sellers get exactly the same expected payoff. Buyers also get zero payoff from this new submarket. The planner's budget constraint is changed in the following way. Originally the planner get the expected payoff of $(\alpha m(\theta_1) + \beta m(\theta_2))h(y) - (\alpha m(\theta_1)p_1 + \beta m(\theta_2)p_1) - k(\alpha\theta_1 + \beta\theta_2)$ from submarkets ϕ_1 and ϕ_2. By creating a new submarket, the planner gets $(\alpha + \beta)m(\theta_3)h(z) - (\alpha + \beta)m(\theta_3)t_{s,3} - k(\alpha + \beta)\theta_3$. The first two terms are exactly the same as in the original case by construction of θ_3 and $t_{s,3}$. But $-k(\alpha\theta_1 + \beta\theta_2)$ is weakly less than $-k(\alpha + \beta)\theta_3$ due to concavity of $m(.)$. Therefore, we can always find another feasible allocation which does not involve pooling or lotteries and creates weakly higher surplus while delivering the planner weakly higher surplus.

I summarize the results discussed so far in the following lemma. In particular, since each type only applies to one submarket, I index submarkets by types of sellers, z.

Lemma 3. *Suppose $(\theta(z), p(z))$solves problem 6 and $t_m(z) = h(z) - p(z) - \frac{k}{q(\theta(z))}$, $T = 0$, $G(z) = \int^z f(\tilde{z})\theta(\tilde{z})d\tilde{z}$ and $H(\phi, z) = \int^{min\{\phi,z\}} f(\tilde{z})d\tilde{z}$. Then (G, H, p, t_m, T) is a constrained efficient allocation.*

Problem 6.

$$\max_{\theta(z),p(z)} \int \Big[m(\theta(z))(h(z) - c(z)) - k\theta(z)\Big]dF$$

$$s.t.\;\; z \in \arg\max_{\hat{z}} U(z, \hat{z}),\;\; \textit{(IC)}$$

$$U(z, z) \geq 0 \;\; \textit{(IR)},$$

$$\textit{and} \;\; \int \Big[m(\theta(z))(h(z) - t_S(z)) - k\theta(z)\Big]dF = 0 \;\; \textit{(BB)},$$

$$\textit{in which } V(z, \hat{z}) \equiv m(\theta(\hat{z}))(p(\hat{z}) - c(z)).$$

To understand this problem, I break down the sellers' maximization problem into two constraints. The first one is Incentive compatibility (IC) constraint, stating that type z should get weakly lower payoff by applying to submarket $\hat{z}$. Formally, let $U(z, \hat{z})$ denote the payoff of type z from applying to submarket $\hat{z}$. IC implies that $U(z, \hat{z}) \leq U(z, z)$. The second one is participation or individual rationality (IR) constraint, stating that all types need to get at least 0 payoff to participate. The last constraint in the problem is planner's budget constraint. I denote it by BB which stands for Budget Balance condition.

In the lemma, definitions of $G(.)$ and $H(., .)$ require explanation. Regarding G distribution, since each type z applies only to submarket z, the measure of buyers at each submarket z is $F'(z)\theta(z)$. To calculate $G(z)$, I just need to integrate over all submarkets with indices less than z. Regarding H distribution, first assume that $\phi \geq z$. Since each type z applies only to submarket z, the measure of types who apply to submarkets less than ϕ is the measure of all types below z. Now assume that $\phi < z$. The measure of types who apply to submarkets less than ϕ is the measure of all types below ϕ.

Step 2: Characterizing Incentive Compatible Schemes

Suppose $c(z)$ is strictly monotone in z. The first two parts of lemma state that $c'(z)\frac{d\theta(y)}{dy} \leq 0$ is necessary and sufficient for $\theta(z)$ to be implementable, that is, there exists some transfers that together with $\theta(z)$ satisfies IC. The third part gives us necessary condition for $U(z)$, where $U(z)$ denotes the payoff that type z gets from some allocation. (I suppress the functionality of $U(.)$ to allocation to reduce the notation.)

Lemma 4 (Necessary and sufficient condition for $\theta(z)$ to be implementable)**.** *Assume that c(z) is strictly monotone in y.*

1. *A piecewise C^1 function $\theta(z)$ is implementable only if $c'(z)\frac{d\theta(z)}{dz} \leq 0$ wherever $\theta(z)$ is differentiable at z.*

2. *Any piecewise C^1 function $\theta(z)$ satisfying $c'(z)\frac{d\theta(z)}{dz} \leq 0$ is implementable.*

3. *If $(a(.), r(.))$ satisfies I.C. constraint, then $U(z)$ must satisfy*

$$U(z) = U(z_H) + \int_z^{z_H} m(\theta(z_0))c'(z_0)dz_0.$$

Proof. $U(z, \hat{z})$ can be written as $U(z, \hat{z}) = x(\hat{z})c(z) + r(\hat{z})$ where $x(\hat{y}) = -m(\theta(\hat{y}))$ and $r(\hat{y}) = m(\theta(\hat{y}))p(\hat{y})$. Following Fudenberg and Tirole (1991), p. 257, I say, formally, that $\theta(z)$ is implementable if there exists a function $r(z)$ such that the allocation $\theta(z), r(z)$ satisfy IC constraint in problem 6. According to Fudenberg and Tirole (1991) theorem 7.1, the **necessary** condition for $x(.)$ to satisfy IC is

$$\frac{\partial}{\partial z}\left(\frac{\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial r}}\right)\frac{dx}{dz} \geq 0.$$

But $\frac{\partial}{\partial z}(\frac{\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial r}})\frac{dx}{dz} = \frac{\partial}{\partial z}(\frac{c(z)}{1})(-m'(\theta(z)))\frac{d\theta(z)}{dz}$. But $c'(.) > 0$ and $m'(.) \geq 0$,therefore, the necessary condition is equivalent to

$$\frac{d\theta(z)}{dz} \leq 0. \tag{30}$$

According to Fudenberg and Tirole (1991) theorem 7.3, the **sufficient** condition for $x(.)$ to be implementable is that $\frac{dx(z)}{dz} \geq 0$, or equivalently, $\frac{d\theta(z)}{dz} \leq 0$. [14]

For the third part of the lemma, I use corollary 1 from Milgrom and Segal (2002), if $\theta(y)$ satisfies IC, then $U(.)$ can be written as follows:

$$U(z) = U(z_H) - \int_z^{z_H} [\frac{\partial U(z_0, z_0)}{\partial z}]dz_0 = U(z_H) - \int_z^{z_H} [-m(\theta(z_0))c'(z_0)]dz_0. \tag{31}$$

This equation is derived from Envelop theorem and is standard in mechanism design literature. The requirements that we need to check are as follows:

1. $U(z, \hat{z})$ is differentiable and absolutely continuous in z. This is satisfied because c is assumed to be twice differentiable.

[14]Briefly, the idea of the proof for necessity is that the second order condition for IC maximization problem ($\max_{\hat{z}} V(z, \hat{z})$) should hold. For sufficiency, the proof goes by contradiction. The proof of this lemma is standard in mechanism design literature thus omitted from here.

2. $\sup_{\hat{z}} \left|\frac{\partial U(z,\hat{z})}{\partial z}\right|$ is integrable. This is satisfied because $\sup_{\hat{z}} \left|\frac{\partial U(z,\hat{z})}{\partial z}\right| \leq \left|c'(z)\right| < M$ for some $M \in \mathbb{R}$, because $c'(.)$ is continuous and is defined over a compact set $[z_L, z_H]$.

3. $\theta(z)$ is obviously non-empty. □

We know that under IC $U(z) = m(\theta(z))(p(z) - c(z))$, for all z. I substitute $U(.)$ from equation 31 to derive the price which satisfies IC:

$$p(z) = c(z) + \frac{U(z_H) + \int_z^{z_H} m(\theta(z_0))c'(z_0)dz_0}{m(\theta(z))}. \tag{32}$$

So far I have reduced IC constraint in the planner's problem to two conditions 30 and 32. Therefore, thanks to Lemma 4, I can rewrite the planner's problem as follows:

Problem 7. *Planner's problem*

$$\max_{\theta(z),p(z)} \int \Big[m(\theta(z))[h(z) - c(z)] - k\theta(z)\Big] F'(z)dz$$

$$s.t.\ \frac{d\theta(z)}{dz} \leq 0,\ p(z) = c(z) + \frac{U(z_H) + \int_z^{z_H} m(\theta(z_0))c'(z_0)dz_0}{m(\theta(z))},$$

$$U(z) \geq 0,\ \ and\ \int \Big[m(\theta(z))[h(z) - t_S(z)] - k\theta(z)\Big] F'(z)dz = 0.$$

From now one, I work with this problem and characterize the solution to this problem.

8.5.2 Proof of Proposition 5 under Part 1 of Assumption 5

I prove Proposition 5 under the first assumption ($h'(.) \leq 0$). In order to solve the planner's problem, I use somewhat a backward approach. I first guess that the planner can achieve the complete information level of welfare. That is, the planner can maximize his objective function point-wise. What I need to do then is to Check that the monotonicity constraint is satisfied. Finally, I need to check that $U(z)$ for all z is non-negative while planner's budget constraint is satisfied.

Proposition 8. *Proposition 5 under part 1 of Assumption 5*

Assume that $c'(.) > 0$ and $h'(.) \leq 0$. Then $\theta(z) = \theta^{CI}(z)$ for all z and

$$U(z_H) = \int \int_z^{z_H} m(\theta(z_0))h'(z_0)F'(z)dz_0dz + U^{CI}(z_H).$$

Also, $p(.)$ is pinned down by equation 32.

Proof. Point-wise maximization yields the following FOC:

$$m'(\theta(z))(h(z) - c(z)) - k = 0. \tag{33}$$

By differentiating 33 with respect to z, one yields

$$\frac{d\theta(z)}{dz} = -\frac{k(h'(z) - c'(z))}{m''(\theta(z))(h(z) - c(z))^2}. \tag{34}$$

By assumption, $h'(z) - c'(z) \leq 0$ and $m''(\theta(z)) \leq 0$, therefore $\frac{d\theta(z)}{dz}$ is negative. Hence, $c'(z)\frac{d\theta(z)}{dz} \leq 0$ constraint is satisfied in problem 7. Now, I calculate that budget constraint in terms of $U(z_H)$ and other variables and equate it to 0. Then I derive $U(z_H)$ and ensure that $U(z_H)$ is positive.

$$\int [m(\theta(z))[h(z) - p(z)] - k\theta(z)] \, F'(z)dz$$

$$= \int [m(\theta(z))[h(z) - c(z)] - k\theta(z) - m(\theta(z))(p(z) - c(z))] \, F'(z)dz$$

$$= \int \left[[-\int_z^{z_H} m(\theta(z_1))[h'(z_1) - c'(z_1)]dz_1 + U^{CI}(z_H)] - \int_z^{z_H} m(\theta(z_0))c'(z_0)dz_0 - U(z_H) \right] F'(z)dz$$

$$= \int \int_z^{z_H} [-m(\theta(z_0))[h'(z_0) - c'(z_0)] - m(\theta(z_0))c'(z_0)\,]F'(z)dz_0dz + U^{CI}(z_H) - U(z_H)$$

$$= -\int \int_z^{z_H} m(\theta(z_0))h'(z_0)F'(z)dz_0dz + U^{CI}(z_H) - U(z_H) = 0 \tag{35}$$

Therefore, if $U(z_H)$ is chosen as follows, the budget constraint holds:

$$U(z_H) = -\int \int_z^{z_H} m(\theta(z_0))h'(z_0)F'(z)dz_0dz + U^{CI}(z_H).$$

Since $U^{CI}(z_H) \geq 0$ and $h' \leq 0$, so $U(z) \geq 0$. Moreover since $c'(z) > 0$, the integral in 32 is positive and consequently, $p(z) \geq c(z)$ is also satisfied, implying that all types get positive payoff. □

The proposition and its proof can be written in the exactly same fashion if instead $c(.)$ is strictly decreasing and $h(.) - c(.)$ is increasing. The result are not reported to save space. Note that in order to show that IC is satisfied, it is sufficient to have $h' - c' \leq 0$ (see equation 34). In the next part of the proposition, I replace the assumption $h' \leq 0$ with this weaker assumption $h' - c' \leq 0$. To satisfy planner's budget constraint, I need another assumption summarized in the second part of Assumption 5.

8.5.3 Proof of Proposition 5 under Part 2 of Assumption 5

Proposition 9. *Suppose the second part of Assumption 5 holds. Then for all z,* $\theta(z) = \theta^{CI}(z)$*. Also,*

$$U(z_H) = \int \left[m(\theta(z))[h(z) - c(z)] - k\theta(z) - m(\theta(z))c'(z)(\frac{F(z)}{F'(z)}) \right] g(z)dz.$$

Prices are pinned down by equation 32.

Proof. The proof is similar to proof of the first part discussed above. Note that $h'(z) - c'(z)$ is negative. According to equation 34, monotonicity constraint (that $\theta(z)$ is decreasing in θ) is satisfied. Also $U(z_H)$ is positive as will be shown below. The integral in 32 is positive because $c'(z) > 0$ and $U(z_H) \geq 0$, therefore all types (not only z_H) get a positive payoff. To check that planner's budget constraint holds, I write the planner's budget in terms of $U(z_H)$ and other variables and equate it to 0. Then I ensure that $U(z_H) \geq 0$. Now I calculate planner's budget:

$$\int [m(\theta(z))[h(z) - p(z)] - k\theta(z)] F'(z)dz$$

$$= \int [m(\theta(z))[h(z) - c(z)] - k\theta(z) - m(\theta(z))(p(z) - c(z))] F'(z)dz$$

$$= \int \left[m(\theta(z))[h(z) - c(z)] - k\theta(z) - \int_z^{z_H} m(\theta(z_0))c'(z_0)dz_0 \right] F'(z)dz - U(z_H)$$

$$= \int \left[m(\theta(z))[h(z) - c(z)] - k\theta(z) - m(\theta(z))c'(z)\frac{F(z)}{F'(z)} \right] F'(z)dz - U(z_H) = 0. \quad (36)$$

The second equality follows equation 32. The third equality is established using integration by parts[15]. From equation 36, I can write

$$U(z_H) = \int \left[m(\theta(z))[h(z) - c(z)] - k\theta(z) - m(\theta(z))c'(z)\frac{F(z)}{F'(z)} \right] F'(z)dz.$$

A sufficient condition for the integral to be positive is that the sum of terms in the brackets is always positive, that is, for all z: $m(\theta(z))[h(z) - c(z)] - k\theta(z) - m(\theta(z))c'(z)(\frac{F(z)}{F'(z)}) \geq 0$. But at the solution $m'(\theta(z))[h(z) - c(z)] = k$, therefore

$$\frac{m(\theta(z))[h(z) - c(z)] - k\theta(z)}{m(\theta(z))}$$

$$= \frac{m(\theta(z))[h(z) - c(z)] - m'(\theta(z))[h(z) - c(z)]\theta(z)}{m(\theta(z))} = -\frac{\theta(z)q'(\theta(z))}{q(\theta(z))}(h(z) - c(z)).$$

[15] For any differentiable functions F and G, if $G(z_H) = 1$, $G(z_L) = 0$, $F(z_H) = 1$ and $F(z_L) = 0$ we will have: $\int_z^{z_H} F(z)g(z)dz = \int_{z_L}^{z_H} F'(z)(1 - G(z))dz$ using integration by parts.

Hence, for the $U(z_H)$ to be positive, it is sufficient to have:

$$\eta(\theta(z))\frac{h(z)-c(z)}{c'(z)} \geq \frac{F(z)}{F'(z)}.$$

From $m'(\theta(z))[h(z)-c(z)] = k$, I can write $\theta(z)$ as the following $\theta(z) = m'^{-1}(\frac{k}{h(z)-c(z)})$. Replacing $\theta(.)$ in the sufficient condition yields $\psi(\frac{k}{h(z)-c(z)})\frac{h(z)-c(z)}{c'(z)} \geq \frac{F(z)}{F'(z)}$ which is the same as the left hand side of the second part of Assumption 5. This concludes the proof. □

When $c'(.) < 0$ and $h'(.) - c'(.) \geq 0$, a similar result can be obtained. We want to understand the condition in the above Assumption better. It's easy to show that if $1/q(\theta)$ is convex, $\eta(\theta)$ is increasing in θ. Also, m' is decreasing in θ by assumption. Hence, $\psi(.)$ is a decreasing function. The second part of Assumption 5 states that for a given distribution, for a given z and a given value for $c'(z)$, $h(z)-c(z)$ should be sufficiently high or k should be sufficiently low. The intuition is that the surplus generated by type z should be sufficiently high (or the entry cost sufficiently low) to provide enough resources for the planner to implement the full information allocation. This assumption is exactly the counterpart of the third part of Assumption 2 in the discrete type case.

8.5.4 Proof of Proposition 6

Proof. By assumption, the complete information case is not achievable. Therefore, the guess-and-verify approach does not work. To solve for planner's problem, consider problem 7. I first ignore monotonicity constraint. Then I form the Lagrangian and derive first order condition (FOC). Then I verify that the monotonicity constraint (and consequently IC) is also satisfied. Denote the Lagrangian by $\mathcal{L}$ and the Lagrangian multiplier by μ:

$$\mathcal{L} = \int \Big[m(\theta(z))(h(z)-c(z)) - k\theta(z)$$

$$+\mu\Big[m(\theta(z))(h(z)-c(z)) - k\theta(z) - m(\theta(z))c'(z)(\frac{F(z)}{F'(z)}) - U(z_H)\Big]\Big]F'(z)dz. \quad (37)$$

To write the Lagrangian, I substituted the price from equation 32 to the BB constraint (see the derivation in the proof of Proposition 9, equation 35). The FOC with respect to $\theta(z)$ for all z is given by:

$$m'(\theta(z))(h(z)-c(z)) - k + \mu(m'(\theta(z))(h(z)-c(z)) - k) - \mu m'(\theta(z))c'(z)\frac{F(z)}{F'(z)} = 0. \quad (38)$$

It can be simplified to conform to equation 3 exactly.

According to the assumptions of the proposition, $h-c$ and $h-c+c'\frac{F(z)}{F'(z)}$ are decreasing in z. Also, μ is non-negative, so $h(z)-c(z)+\frac{\mu}{1+\mu}c'(z)\frac{F(z)}{F'(z)}$ is also decreasing in z. Therefore, FOC implies that $\theta(z)$ is decreasing in z as well. As a result, the monotonicity constraint ($c'(z)\frac{d\theta(z)}{dz}\leq 0$) is satisfied. The complete information allocation is not achievable, so if the planner allocates all types the market tightness $\theta^{CI}(z)$ (and the corresponding transfers from equation 32), then the derived value for $U(z_H)$ becomes negative. (otherwise the previous Proposition 9 implies that the complete information allocation can be achieved).

Assume there exists a $\mu>0$ that the FOC and BB both hold. Since the objective function is strictly concave in $\theta(.)$, and the objective function is just the sum of concave functions, the objective function is also concave in $\theta(.)$[16].Because of concavity of the objective function, the FOC is sufficient for the solution. Hence, it only remains to show that such a $\mu>0$ exists.

Note that $\theta(.)$ obtained from equation 38 is continuous in μ. Accordingly, the LHS of the equation is continuous in μ as well. I need to show that the LHS is negative at $\mu=0$ and is positive when $\mu\to\infty$.

If $\mu=0$, then $\theta(z)=\theta^{CI}(z)$. Because complete information allocation is not achievable, $\int\Big[m(\theta^{CI}(z))\Big[h(z)-c(z)-c'(z)\frac{F(y)}{f(y)}\Big]-k\theta(z)\Big]g(z)dz<0$. (Otherwise, according to Proposition 5, I can find a set of transfers such that the complete information allocation is achievable.)

If $\mu\to\infty$, then $m'(\theta(y))\left[h(z)-c(z)-c'(z)\frac{F(z)}{F'(z)}\right]=k$. Therefore,

$$\int\Big[m(\theta(z))\Big[h(z)-c(z)-c'(z)\frac{F(z)}{F'(z)}\Big]-k\theta(z)\Big]g(z)dz$$

$$=\int[m(\theta(z))\frac{k}{m'(\theta(z))}-k\theta(z)]g(z)dz=k\int\frac{m(\theta(z))-\theta m'(\theta(z))}{m'(\theta(z))}g(z)dz$$

$$=-k\int\frac{\theta(z)^2q'(\theta(z))}{m'(\theta(z))}g(z)dz>0,$$

where the first equality is derived from the FOC when $\mu\to\infty$ and the last inequality follows

[16] To show this point, consider a simpler version where the objective is a function of two variables, that is, $g(x_1,x_2)=f(x_1)+h(x_2)$. Also assume $f(.)$ and $h(.)$ are concave in x_1 and x_2 respectively. I want to show that g is concave in (x_1,x_2). TO show that, I form the Hessian as follows:

$$\begin{bmatrix} f'' & 0 \\ 0 & h'' \end{bmatrix}$$

Since f'' and h'' are both negative, the determinant of Hessian is negative. Therefore g is concave.

from $q' < 0$ and $m' > 0$. According to intermediate value theorem, there exists a strictly positive μ which satisfy the above requirement. $\square$

Example 1. *Model parameters:* $m(\theta) = 1 - e^{-\theta}$, $Z = [z_L, z_H] \subset R_{++}$, $c(z) = \beta z$, $h(z) = \beta z + s$ *where* $\beta > 0$. $f(.)$ *is log-concave (like uniform, truncated normal and many popular distributions), so* $\frac{1-F}{f}$ *is decreasing in* z. *Then BB is satisfied if* $\frac{s-k(1-ln(\frac{k}{s}))}{s-k}\frac{s}{\beta} > \frac{1}{f(z_L)}$. $p^{CI}(z) = \eta(m^{-1}(\frac{k}{s}))h(z) + (1 - \eta(m^{-1}(\frac{k}{s})))c(z) = \psi(\frac{k}{s})s + c(z)$. *The market tightness at the constrained efficient allocation is given by* $\theta(z) = \theta^{CI}(z) = m^{-1}(\frac{k}{s}) = ln(\frac{s}{k})$.

References

D. Acemoglu and R. Shimer. Holdups and efficiency with search frictions. *International Economic Review*, 40(4):827–849, 1999.

G. Akerlof. The economics of caste and of the rat race and other woeful tales. *The Quarterly Journal of Economics*, pages 599–617, 1976.

G. A. Akerlof. The market for" lemons": Quality uncertainty and the market mechanism. *The quarterly journal of economics*, pages 488–500, 1970.

B. Chang. Adverse selection and liquidity distortion. *Available at SSRN 1701997*, 2012.

A. Delacroix and S. Shi. Pricing and signaling with frictions. *Journal of Economic Theory*, 148(4):1301–1332, 2013.

J. Eeckhout and P. Kircher. Sorting and decentralized price competition. *Econometrica*, 78 (2):539–574, 2010.

D. Fudenberg and J. Tirole. Game theorymit press. *Cambridge, MA*, 1991.

V. Guerrieri. Inefficient unemployment dynamics under asymmetric information. *Journal of Political Economy*, 116(4):667–708, 2008.

V. Guerrieri and R. Shimer. Dynamic adverse selection: A theory of illiquidity, fire sales, and flight to quality. *American Economic Review*, 104(7):1875–1908, 2014.

V. Guerrieri, R. Shimer, and R. Wright. Adverse selection in competitive search equilibrium. *Econometrica*, 78(6):1823–1862, 2010.

A. J. Hosios. On the efficiency of matching and related models of search and unemployment. *The Review of Economic Studies*, 57(2):279–298, 1990.

J.-J. Laffont and D. Martimort. *The theory of incentives: the principal-agent model*. Princeton University Press, 2009.

P. Milgrom and I. Segal. Envelope theorems for arbitrary choice sets. *Econometrica*, 70(2): 583–601, 2002.

P. Milgrom and C. Shannon. Monotone comparative statics. *Econometrica: Journal of the Econometric Society*, pages 157–180, 1994.

E. R. Moen. Competitive search equilibrium. *Journal of Political Economy*, 105(2):385–411, 1997.

E. R. Moen and Å. Rosén. Incentives in competitive search equilibrium. *The Review of Economic Studies*, page rdq011, 2011.

D. T. Mortensen and C. A. Pissarides. Job creation and job destruction in the theory of unemployment. *The review of economic studies*, 61(3):397–415, 1994.

D. T. Mortensen and R. Wright. Competitive pricing and efficiency in search equilibrium*. *International Economic Review*, 43(1):1–20, 2002.

S. Shi. Frictional assignment. i. efficiency. *Journal of Economic Theory*, 98(2):232–260, 2001.

S. Shi. A directed search model of inequality with heterogeneous skills and skill-biased technology. *The Review of Economic Studies*, 69(2):467–491, 2002.

R. Shimer. The assignment of workers to jobs in an economy with coordination frictions. *Journal of Political Economy*, 113(5), 2005.