Bargaining and Buyout*

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Abstract

We introduce a noncooperative coalitional bargaining model for characteristic function form games. A player not only *buys out other players' resources and rights* with upfront transfers as in Gul (1989), but also *strategically chooses partners* instead of bargaining with a randomly selected opponent. Such transactions among players are interpreted as coalition formation. The main theorem provides a general inefficiency result. If a characteristic function form game has a strict subcoalition with a strictly positive worth and a player with a strictly positive marginal contribution to the grand-coalition, then an efficient stationary subgame perfect equilibrium does not exist, as long as the discount factor is sufficiently high but strictly less than 1.

Two special results are established. A grand-coalition equilibrium is impossible when players are sufficiently patient, unless the characteristic function form game is a unanimity game. For a simple game with a veto player and multiple winning coalitions, a non-minimal winning coalition is formed with positive probability. In two applications, we study players' strategic alliance behavior and the effect of the strategic behavior on inequality. First, for three-player simple games, the equilibrium payoff vector *Lorenz-dominates* both the Shapley-Shubik power index and the core-constrained Nash bargaining solution. Second, for wage bargaining games, workers endogenously form a union and their equilibrium payoffs can be greater than marginal products.

keywords: noncooperative bargaining, strategic coalition formation, buyout, efficiency, inequality, simple games, weighted majority games, wage bargaining.

JEL Classification: C72, C78; D72, D74

1 Introduction

When three or more players bargain over their joint surplus, forming a transitional coalition is pervasive though such a coalition is inefficient. Rather than immediately forming an efficient coalition, players can increase their bargaining power through a transitional inefficient coalition. In wage bargaining, for instance, workers form a labor union even though the union itself produces nothing.

*For the most recent version, please visit my website: https://sites.google.com/site/joosungecon/. My job market papers consist of two companion papers. This paper deals with general characteristic function form games, and the second one, *"Noncooperative Unanimity Games in Networks,"* extends the idea of this paper to network-restricted games.

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Similarly, in legislative bargaining, minor parties form a coalition though the coalition is still minor. In many coalitional bargaining models, however, players immediately form an efficient coalition, especially when the gain from cooperation is substantial and commonly known.[1]

To investigate players' strategic alliance behavior and gradual agreement phenomena, we introduce a noncooperative bargaining model with *buyout options* for cooperative characteristic function form games. By buyout options we mean, each player can buy out other players' resources and rights with upfront transfers. Players who sell their resources and rights leave the game with receiving monetary transfers; and players who buy out other players remain in the game. As in Gul (1989), we interpret such transactions among players as coalition formation. That is, a player forms a coalition with other players whom she has previously bought out, and then she can exercises all the resources and the rights on the coalition thereafter. Thus, she exclusively derives a surplus from the coalition according to a characteristic function, and she can also sell it out later on.

To be concrete, a noncooperative coalitional bargaining game proceeds as follows. In each period, a proposer is randomly selected according to a given recognition probability. The proposer makes an offer specifying a coalition to bargain and monetary transfers to each member in the coalition. If all the members in the coalition accept the offer, then the proposer remains in the game, inheriting other respondents' resources and rights. Then each remaining player derives a per-period payoff from the coalition which *belongs to* the player.

1.1 Preview of Results

General Inefficiency Results

When players have buyout options, they not only consider the surplus from the current coalition, but also take into account their bargaining power in the subsequent bargaining game. This additional strategic consideration may hinder efficiency. It turns out that an efficient equilibrium is generically impossible, if the discount factor is sufficiently high but strictly less than 1. If a characteristic function form game has a strict subcoalition with a strictly positive worth and a player with a strictly positive marginal contribution to the grand-coalition, then an efficient stationary subgame perfect equilibrium does not exist as long as the discount factor is high enough.[2] The interesting point is that a subcoalition whose worth is zero can be formed in equilibria, because players can improve their future bargaining power by forming a transitional coalition even though it produces nothing. If only a grand-coalition has a strictly positive worth, that is a unanimity game, then any stationary subgame perfect equilibrium is efficient and the equilibrium payoff vector is unique for *all* discount factors.

Two special results are established. The first one is about grand-coalition equilibria, in which all the players always immediately form a grand-coalition. We show that a grand-coalition equilibrium is impossible if the discount factor is close to 1, unless it is a unanimity game. That is, even if the

[1]With incomplete information, delay in an equilibrium is a common feature. With complete information, delayed equilibria have been studied in Chatterjee et al. (1993); Cai (2000), but those equilibria rely on some restrictive bargaining protocols and some specific characteristic functions.

[2]Our result says that inefficiency occurs for high discount factors, however it disappears as the discount factor converges to 1. See Figure 2 in Section 7.1.

gain from forming the grand-coalition is substantial like convex games, the grand-coalition will not be immediately formed with positive probability and hence delay may occur in any equilibrium.

The second special result is for simple games, in which only winning coalitions generates a unit surplus. For simple games with a veto player and multiple winning coalitions, non-winning coalitions may be formed as a transitional state and the final winning coalition is not necessarily minimal, unless all the players are veto. That is, non-veto players may form a coalition each other in an equilibrium, even though it is not a winning coalition. Furthermore, the equilibrium payoff vector is not necessarily in the core of the underlying characteristic function form game.

Reduction of Inequality – Two Applications

As a first application, three-player simple games are studied. Either if there is no veto player or if all the players are veto, then a winning coalition is always immediately formed and each player's equilibrium payoff is $\frac{1}{3}$ of the total surplus. When there is only one veto player, the other non-veto players form a coalition each other with positive probability to become a new veto player in the subsequent two-player game. If there are two veto players, each veto player may form a non-winning coalition with a non-veto player as a transitional state to oppose the other veto player. For each case, weighted-majority games are considered as specific examples.

Interestingly, the equilibrium payoff vector Lorenz-dominates both the Shapley-Shubik power index and the core-constrained Nash bargaining solution. Note that the Shapley-Shubik power index (Shapley and Shubik, 1954) is implemented (except for a unanimity game) when bilateral meetings are randomly selected as in Gul (1989), and the core-constrained Nash bargaining solution is implemented when they have no buyout option as in Okada (1996). Thus, if players have buyout options or if they can strategically choose a coalition, then the inequality of the equilibrium payoffs can be reduced.

We also investigate workers' strategic behavior on union formation in a simple wage bargaining game. Workers endogenously form a union with a positive probability as long as the common discount factor is high enough. That is, each worker buys out the other worker to unify the negotiation channel to the firm rather than bargaining directly with the firm. The rolr of recognition probabilities is also studied. As the workers' recognition probability decreases, the equilibrium wage also decreases but workers form a union more likely.

Especially when the marginal product of labor is low, the equilibrium wage is strictly greater than the marginal product due to workers' strategic alliance behavior. This is contrast to Okada (2011)'s result, in which players have no buyout option. Without buyout options, each worker directly bargains with the firm rather than forming a union, and hence the equilibrium wage converges to their marginal product as a discount factor closes to 1. We also compare the equilibrium payoff vector to the other cooperative solution concepts, including the Shapley value, nucleolus, and the core-constrained Nash bargaining solution. (See Figure 4 in Section 7.3.)

1.2 Related Literature

Noncooperative Coalitional Bargaining Models

The notion of buyout in multi-player bargaining is from Gul (1989) and we succeed his model.[3] Both his model and our model view coalition formation as trading resources and the player who buys out other players upfront can exercises all the resources and the rights on the coalition thereafter. However, his model assumes random meetings, that is, players cannot choose partners to bargain with and their strategic decision is limited on splitting the joint surplus in a randomly selected bilateral meeting.[4] However, as Hart and Mas-Colell (1996) pointed out, a random-meeting-model does not entirely capture players' strategic behavior and the issue about strategic decision on coalition formation has been remained as an open question.

Most of noncooperative bargaining models do not allow players to buy out other players. For instance, Selten (1988) and Compte and Jehiel (2010) assume *one-stage property*, that is, the game is terminated right after any coalition is formed. Other models including Baron and Ferejohn (1989), Chatterjee et al. (1993), and Okada (1996, 2011) assume *exclusion property*; once players form a coalition, all the player in the coalition must exit the game and they are excluded in the further bargaining game. In such environments, players have at most one chance to form a coalition. Thus, they concern the surplus from the current coalition, but not the future coalitional structure which induced by the coalition formation. With exclusion property, inefficient coalitions or transitional coalitions are hardly formed in an equilibrium. For instance, a grand-coalition is always immediately formed for convex games; only minimal coalitions are formed for simple games; and subcoalitions whose surplus is zero never formed. By allowing buyout options, we can investigate players' strategic alliance behavior and gradual agreement phenomena.

Our model is close to Seidmann and Winter (1998), Okada (2000) and Gomes (2005), which allow *renegotiations* in coalitional bargaining.[5] Seidmann and Winter (1998) considered the rejector-proposer model with semi-strict superadditive games; while our model is based on a random-proposer model and cover more general class of games including the cases with multiple efficient coalitions. Okada (2000) considers renegotiations in a random-proposer model. However, he assumes that once a subcoalition formed, any other disjoint subcoalition cannot be formed and the subsequent bargaining is limited to a renegotiation with the previously formed coalition. Gomes (2005) introduces a renegotiation model with externalities, namely a multilateral contracting game, for partition function form games. In a multilateral contracting game, players can make a contract contingent on the other players' coalitional structure.[6] However, in our model, players trade their resources and rights with upfront transfers, which do not depend on future events. That is, our contribution is to explore

[3]Krishna and Serrano (1995) also adopt the notion of buyout to refine a unique subgame perfect equilibrium in unanimity games.

[4]This random meeting assumption simplifies the analysis and hence it is widely accepted especially for games in networks. For instance, Manea (2011b,a); Abreu and Manea (2012a,b) assume bilateral random meetings and Nguyen (2012) imposes multilateral random meetings. Our companion paper, Lee (2013), allows both strategic coalition formation and buyout option in noncooperative network-restricted games.

[5]In *real-time* framework, Hyndman and Ray (2007) introduce a noncooperative coalitional bargaining model with *binding agreements*.

[6]Gomes and Jehiel (2005) allow that coalitions may break up.

inefficiency caused by players' strategic behavior without relying on externalities across coalitions.[7]

In the renegotiation models, once a coalition is formed, it behaves as a single player but all the players in the coalition remain in the game. Thus, the probability that a coalition is recognized as a proposer is the sum of individual recognition probabilities of the players in the coalition. This corresponds to an environment in which players can trade their chances to be a proposer in addition to their resources. We consider such an environment with transferable recognition probabilities as a benchmark case, but we also consider an alternative environment in which players cannot trade their recognition probabilities. In fact, both environments are introduced by Gul (1989). He assumes that the initial recognition probabilities are uniform over players and they cannot trade their recognition probabilities. Thus in any subgame, all the remaining players are selected as a proposer equally likely. He also considers a namely *partnership game* in which players can trade their recognition probabilities and argues that non-transferability of recognition probabilities is essential to implement the Shapley value.

Efficiency and Equality in Coalitional Bargaining

Efficiency has been a central question in bargaining literature. Rubinstein (1982) shows the unique subgame perfect equilibrium yields an immediate agreement in his seminar two-player alternating-offer bargaining model. There are mainly two strands of literature that follow Rubinstein (1982). The first one is two-player bargaining game with incomplete information, which causes delayed and inefficient equilibria. The second one is multi-player bargaining game. In this paper, we focus on multi-player bargaining with complete information.

In coalitional bargaining literature, Chatterjee et al. (1993) provide a condition for efficient stationary subgame perfect equilibria in a rejector-proposer model. That is, if a underlying characteristic function form game is *dominated by its grand coalition*, that is, the grand-coalition has the largest value per capita among all coalitions, then the grand-coalition is always immediately formed for a sufficiently large discount factor. It turns out that this condition is robust in the selection rule of proposers.[8] Thus, this result suggests that efficiency can be obtained when the gain from the grand-coalition is substantial. Our main result implies, however, inefficiency is more pervasive when players have buyout options. If players can trade their recognition probabilities and they are sufficiently patient, then they may form inefficient coalitions, unless the underlying game is a unanimity game. Even if players cannot trade their recognition probabilities, it requires a stronger condition than domination by a grand-coalition for an efficient equilibrium to be exist.

In our model, inefficiency emerges when the discount factor is sufficiently large, because players form an inefficient coalition as a transitional state. However, efficient coalitions will be formed within a finite period. Therefore, with respect to efficiency, there may exist an efficient equilibrium for

[7] Gomes (2005) also defined a coalitional bargaining game as a tentative step to show the existence of equilibria in a multilateral contracting game. Though he did pay little attention to the coalitional bargaining game itself, except for showing the existence of equilibria, however, in terms of a model, his coalitional bargaining game is essentially the same as the idea of buyout options with transferable recognition probabilities.

[8] See Okada (1996) for a random-proposer model. Ray and Vohra (2013) also generalizes this result to a model that combines a rejector-proposer model and a random-proposer model.

a sufficiently low discount factor, and inefficiency asymptotically disappears as the discount factor converges to 1. Such aspects of efficiency have been studied in various renegotiation models. Seidmann and Winter (1998) and Okada (2000) shows that an efficient equilibrium exists for sufficiently low discount factors.[9] For asymptotic efficiency, Seidmann and Winter (1998), Okada (2000) and Gomes and Jehiel (2005) show that efficient coalitions are obtained in a finite period and hence the equilibrium outcome is *almost Pareto efficient* as a discount factor converges to 1. Hyndman and Ray (2007) investigate this asymptotic efficiency in a continuous time framework.

In addition to efficiency, equality has been an important issue especially in normative aspects of cooperative bargaining problems. In noncooperative bargaining literature, Chatterjee et al. (1993) shows that if the underlying game is strictly convex, then the equilibrium payoff vector converges to the core-constrained Nash bargaining solution, which Lorenz-dominates every other core allocations (Dutta and Ray, 1989).[10] For strictly convex games, this result is quite robust so that the limiting payoff vector coincides with the core-constrained Nash bargaining solution, independently on bargaining protocols and section rules of proposers.

For other than convex games, many researchers have concentrated on simple games especially in the legislative bargaining literature after Baron and Ferejohn (1989). For simple games with veto players, the core allocations may not have a desirable property with respect to equality.[11] Furthermore, as Winter (1996) shows in a random-proposer model, the equilibrium payoff vector lies in the core and hence the veto players take all the surplus. However, most of well-known cooperative power indices, including Shapley and Shubik (1954) power index and Banzhaf (1964) power index, do not necessarily lie in the core and the indices assign even a non-veto player a positive value. Our model has a similar feature. Even a non-veto player gets a positive payoff and it sometime coincides to the Shapley and Shubik (1954) power index. Thus, one can view allowing buyout options as a noncooperative foundation of the cooperative power indices.

Aumann and Myerson (2003) informally argued that forming a non-minimal winning coalition could be stable for some simple games. Their insight is consistent with our result. In our model, since players can participate in the subsequent bargaining game after forming a non-winning coalition, a non-minimal winning coalition may occur in the final state. This result is contrast to existing noncooperative models for simple games, including Montero (2002), Morelli and Montero (2003), Montero (2006) and Montero and Vidal-Puga (2011), in which only a minimal winning coalition occurs in stationary subgame perfect equilibria.

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[9] In the rejector-proposal model with renegotiations, Seidmann and Winter (1998) show that any game has an *immediate coalition formation solution* for sufficient low discount factors (Proposition 2), and a unanimity game has it for any discount factor (Proposition 1). In Okada (2000), he shows that a *renegotiation-proof solution* exists for sufficiently low discount factors (Theorem 3). This theorem also implies that only a unanimity game has a renegotiation-proof solution for all discount factor and vise versa.

[10] This generalizes Rubinstein (1982)'s noncooperative foundation of the Nash bargaining solution into multi-player bargaining problems.

[11] If there is no veto player, then the core is empty unless it is unanimity game. In such games, the limiting equilibrium payoff vector depends on their initial recognition probabilities. Instead of the core-constrained Nash bargaining solution, the relation between the equilibrium payoff vector and the nucleolus (or the kernel) has been studied by Serrano (1993), Serrano (1995), Montero (2002), and Montero (2006).

The paper is organized as follows. Section 2 describes a noncooperative coalitional bargaining model with buyout options. In Section 3, we define a stationary subgame perfect equilibrium and characterize a cutoff strategy equilibrium which is a special form of stationary subgame perfect equilibrium. Section 4 characterizes an existence condition for efficient equilibria. In Section 5, we consider an alternative environment in which players cannot trade their chances to be a proposer. In Section 6 and Section 7, three-player simple games and wage bargaining games are studied as applications. Section 8 concludes this paper with further research issues. Omitted Proofs appear in Appendices.

2 A Model

Let N be a set of players and $v : 2^N \setminus \{\emptyset\} \to \mathbb{R}^N_+$ be a *characteristic function*. We assume that v is zero-normalized, essential, and superadditive, that is, $v(\{i\}) = 0$ for all $i \in N$; $v(N) > 0$; and $v(S \cup S') \geq v(S) + v(S')$ for all $S, S' \subseteq N$ such that $S \cap S' = \emptyset$. A characteristic function v is *monotone* if, for all $S' \subseteq S \subseteq N$, $v(S') \leq v(S)$. Note that monotonicity is implied by superadditivity.

A tuple (N, v) is an *underlying characteristic function form game*, or shortly an *underlying game*. We follows Gul (1989)'s interpretation. Each player initially has a specific resource. Each coalition represents a combination of resources that initially belong to the players in the coalition which generates a flow of surplus according to the characteristic function.

A (coalitional) *state* π is a partition of N specifying a set of *active player*s $N^\pi \subseteq N$. That is, each active player $i \in N^\pi$ *owns* her partition block $[i]_\pi$; and for all $j \in N$, the partition block $[j]_\pi$ *belong*s to only one active player i such that $[j]_\pi \cap N^\pi = \{i\}$. Denote π° by the initial state, that is, $N^{\pi^\circ} = N$ and $[i]_{\pi^\circ} = \{i\}$ for all $i \in N$. A state π is *terminal* if $\sum_{i \in N^\pi} v([i]_\pi) = v(N)$. That is, in any terminal state, there is no unrealized surplus. Let Π be a set of all states.

For each $\pi \in \Pi$ and $i \in N^\pi$, denote $N^\pi_i = \{S \subseteq N^\pi \mid i \in S\}$. For each $i \in N^\pi$ and $S \subseteq N^\pi_i$, i's *S-formation* yields a subsequent state $\pi(i, S)$, where $N^{\pi(i,S)} = (N^\pi \setminus S) \cup \{i\}$, $[i]_{\pi(i,S)} = \cup_{k \in S}[k]_\pi$, and $[j]_{\pi(i,S)} = [j]_\pi$ for all $j \in N^\pi \setminus \{i\}$. For a pair of coalitional states π' and π, π' *succeeds* π, if there exists a sequence of formations $\{(i_\ell, S_\ell)\}^L_{\ell=1}$ such that $\pi' = \pi(i_1, S_1) \cdots (i_L, S_L)$; $i_1 \in N$ and $S_1 \subseteq N_{i_1}$; and $i_\ell \in N^{\pi(i_1,S_1)\cdots(i_{\ell-1},S_{\ell-1})}$ and $S_\ell \subseteq N^{\pi(i_1,S_1)\cdots(i_{\ell-1},S_{\ell-1})}_{i_\ell}$ for all $\ell = 2, \cdots, L$. Given $\pi \in \Pi$, let $\Pi_{|\pi}$ be a set of *succeeding state*s of π.

A *noncooperative coalitional bargaining game*, or shortly, a *bargaining game* is a tuple $\Gamma = (N, v, p, \delta)$, where $p \in \mathbb{R}^N_{++}$ is the *initial recognition probability* with $\sum_{i \in N} p_i = 1$, and $0 < \delta < 1$ is the common discount factor. For each $\pi \in \Pi$, we define the *induced recognition probability* p^π in the following way:

$$p^\pi_i = \begin{cases} \sum_{j \in [i]_\pi} p_j & \text{if } i \in N^\pi \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

That is, if a player forms a coalition, then the player takes other players' recognition probabilities as well. One can view this as forming a partnership.[12]

A bargaining game proceeds as follows.[13] In period t, they begins with the previous state π^{t-1}.

[12] This is a generalization of the *partnership game* which is discussed in Gul (1989).

[13] See Figure 1 for a part of a game tree.

If π^{t-1} is a terminal state, then only a production stage occurs without bargaining stages and hence $\pi^t = \pi^{t-1}$. If π^{t-1} is a non-terminal state, then the period consists of three bargaining stages and one production stage. Each stage is defined as follows:

i) *Recognition:* Nature selects a player $i \in N^{\pi^{t-1}}$ as a proposer with probability $p_i^{\pi^{t-1}}$.

ii) *Proposal:* The proposer i chooses a pair (S, y) of a coalition $S \subseteq N_i^{\pi^{t-1}}$ and monetary transfers $\{y_j\}_{j\in S}$ with $\sum_{j\in S} y_j = 0$.

iii) *Response:* By a given order, each respondent $j \in S \setminus \{i\}$ sequentially either accepts the offer or rejects it. If any $j \in S \setminus \{i\}$ rejects then the current state does not change and hence $\pi^t = \pi^{t-1}$. If all $j \in S \setminus \{i\}$ accept the offer, then the current state transitions to $\pi^t = \pi^{t-1}(i, S)$, that is, each $j \in S \setminus \{i\}$ leaves the game with receiving y_j from the proposer i.

iv) *Production:* Each partition block generates a surplus to the owner. That is, each active player $i \in N^{\pi^t}$ derives $(1-\delta)v([i]_{\pi^t})$.[14]

Given a sequence of states $\tilde{\pi} = \{\pi^t\}_{t=0}^{\infty}$ and a sequence of transfers among players $\tilde{y} = \{y^t\}_{t=0}^{\infty}$, player i's discounted sum of payoffs is

$$U_i(\tilde{\pi}, \tilde{y}) = \sum_{t=1}^{\infty} \delta^{t-1} \left((1-\delta)v([i]_{\pi^t}) + y_i^t\right) \mathbb{1}(i \in N^{\pi^t}).$$

When there is no danger of confusion, we omit π° in notations, for instance, $N^{\pi^\circ(i,S)} = N^{(i,S)}$, $p^{\pi^\circ(i,S)} = p^{(i,S)}$, and so on. For notational simplicity, for any $z \in \mathbb{R}^n$ and $S \subseteq N$, denote $z_S = \sum_{j\in S} z_j$. For a characteristic function v, denote $\bar{v} = v(N)$, $v_i = v(\{i\})$, and $v_S = \sum_{i\in S} v_i$.

3 Stationary Subgame Perfect Equilibria

Our equilibrium concept is a stationary subgame perfect equilibrium (SSPE). A stationary strategy depends only on the current state and *within-period* histories, but not the histories of past periods. Even in the class of stationary subgame perfect equilibria, players' strategies may depend on within-period histories, which involve the identity of the proposer, the proposed coalition and the proposed allocation, preceding respondents' reactions, and so on. In noncooperative coalitional bargaining literature, a *cutoff strategy* is widely used as a special form of stationary strategies. In addition to its tractability, it is known that a cutoff strategy equilibrium is equivalent to a general SSPE in terms of equilibrium payoffs, due to Yan (2003) and Eraslan and McLennan (2013). Through a cutoff strategy equilibrium, the existence of SSPE is also proved by Eraslan (2002), Gomes (2005), and Eraslan and McLennan (2013).

In this section, as a preliminary step, we formally describe an extensive form game in which players have buyout options. Then, we show the payoff equivalence between a cutoff strategy equilibrium and an SSPE, and characterize a cutoff strategy equilibrium with two tractable conditions, optimality and consistency. Note that this section is largely based on Eraslan and McLennan (2013) and Yan (2003).

[14] The coefficient $1-\delta$ normalizes the discounted sum of streams of surplus. Thus, a coalition S generates $(1-\delta)v(S)$ for each period so that the sum of streams of surplus is $v(S) = (1-\delta)v(S) + \delta(1-\delta)v(S) + \delta^2(1-\delta)v(S) + \cdots$.

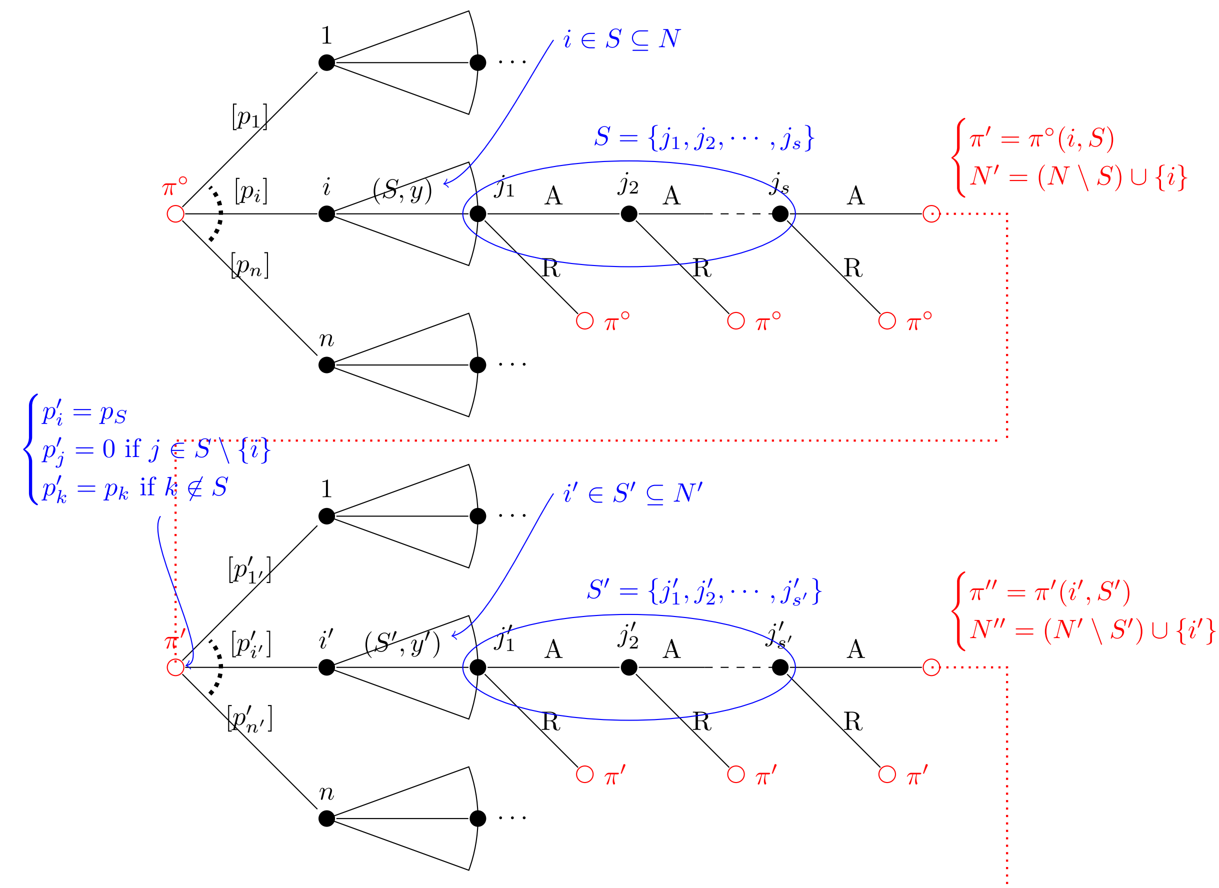


Figure 1: A partial game tree.

3.1 Stationary Strategies

Fix $\Gamma = (N, v, p, \delta)$ and let n be a cardinality of N. Also let $\mathcal{S} = \{S \subseteq N \mid S \neq \emptyset\}$, $\mathcal{S}_i = \{S \subseteq N \mid i \in S\}$ and $X = \mathbb{R}^n_+$. First, we consider the initial state π°. Since a proposer is selected at the beginning of each period, each within-period history always specifies a current proposer. Let $H^0 = N \times \mathcal{S} \times X$ be a set of histories right after the proposer makes an offer; and for all $i \in N$, $H^i = H^{i-1} \times \{0, 1\}$ be a set of histories right after player i responses. The set of all possible *within-period histories* is

$$H = N \cup H^0 \cup H^1 \cup \cdots \cup H^n.$$

For all $h \in H$, typically denoted by $h = (\phi, S, y, r_1, r_2, \cdots, r_i)$, the first element of the history specifies the current proposer, denoted by $\phi(h) = \phi \in N$. For all $h \in \cup_{\ell=0}^n H^\ell$, the second element and the third element specify the proposed coalition and the proposed allocation, denoted by $S(h) = S \in \mathcal{S}$ and $y(h) = y \in X$. For all $i \in N$ and all $h \in \cup_{\ell=i}^n H^\ell$, the $(i+3)$th element specifies the player i's response, denoted by $r_i(h) = r_i \in \{0, 1\}$, where 0 represents i's rejection and 1 represents i's acceptance.

For a measurable space $(\Omega, \mathcal{A})$, let $\Delta(\Omega)$ be the set of probability measures on Ω. For a pair of measurable spaces $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$, let $\Delta(\Omega_1, \Omega_2)$ be the set of transition probabilities from Ω_1 to Ω_2. Player i's *(stationary) proposal strategy* in the initial state is $\alpha_i \in \Delta(\mathcal{S}_i \times X)$. Define a *proposal transition probability* $\alpha \in \Delta(N, \mathcal{S} \times X)$ so that $\alpha(i)(S, y) = \alpha_i(S, y)$. Player i's *(stationary) response strategy* in the initial state is $\beta_i \in \Delta(H^{i-1}, \{0, 1\})$ such that $\beta_i(h)(1) = 1$ if either $\phi(h) = i$ or $i \notin S(h)$.

The three stochastic components, the recognition probability p, the proposal transition probability α, and the response strategy profile $\beta \equiv \{\beta_i\}_{i \in N}$, induce a unique probability measure on H^n.[15] The induced probability measure is denoted by $p \otimes \alpha \otimes \beta_1 \otimes \cdots \otimes \beta_n$.

Let $\mathcal{O} = (\mathcal{S} \times X) \cup \{\pi^\circ\}$ be the outcome space in π°. Define an outcome function $o : H^n \to \mathcal{O}$, such that for all $h \in H^n$,

$$o(h) = \begin{cases} (S(h), y(h)) & \text{if } \times_{j \in N} r_j(h) = 1 \\ \pi^\circ & \text{otherwise.} \end{cases}$$

Let $(p \otimes \alpha \otimes \beta_1 \otimes \cdots \otimes \beta_n) \circ o^{-1}$ be the induced measure on $\mathcal{O}$ by (p, α, β). We also define the induced measure κ on $\mathcal{O}$ by any history $h \in H$ and the initial-state stationary strategy profile (α, β):

$$\kappa(h, \alpha, \beta) = \begin{cases} (\delta_h \otimes \alpha \otimes \beta_1 \otimes \cdots \otimes \beta_n) \circ o^{-1} & \text{if } h \in N \\ (\delta_h \otimes \beta_{\ell+1} \otimes \cdots \otimes \beta_n) \circ o^{-1} & \text{if } h \in H^\ell \quad \ell = 0, 1, \cdots, n, \end{cases}$$

where δ_h is the Dirac probability measure on H.

Let $\mathbf{x}$ be a value profile, that is, $\mathbf{x} = \{\{x_j^\pi\}_{j \in N^\pi}\}_{\pi \in \Pi}$. Given an initial-state stationary strategy profile (α, β) and a value profile $\mathbf{x}$, player i's expected payoffs at $h \in H$ is:

$$\begin{aligned} w_i(h, \alpha, \beta, \mathbf{x}) \quad = \quad & \kappa(h, \alpha, \beta)(\pi^\circ) x_i \\ & + \sum_{S \in \mathcal{S}_i} \int_{y \in X} \left(y_i + \left(x_i^{(i,S)} - \sum_{j \in S} y_j \right) \mathbb{1}(i = \phi(h)) \right) \kappa(h, \alpha, \beta)(S, dy) \\ & + \sum_{S \in \mathcal{S} \setminus \mathcal{S}_i} x_i^{(\phi(h), S)} \int_{y \in X} \kappa(h, \alpha, \beta)(S, dy). \end{aligned}$$

Then an initial-state stationary strategy profile (α, β) is an *initial-state stationary subgame perfect* $\mathbf{x}$*-equilibrium* if, for all $h \in H$, all $i \in N$, and i's all possible initial-state stationary strategies $\hat{\alpha}_i$ and $\hat{\beta}_i$,

$$w_i(h, \alpha, \beta, \mathbf{x}) \geq w_i(h, (\hat{\alpha}_i, \alpha_{-i}), (\hat{\beta}_i, \beta_{-i}), \mathbf{x}). \tag{2}$$

A stationary strategy profile specifies all the active players' initial-state stationary strategies for each subgame. Let $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be a stationary strategy profile, where $\boldsymbol{\alpha} \equiv \{\alpha^\pi\}_{\pi \in \Pi}$ and $\boldsymbol{\beta} \equiv \{\beta^\pi\}_{p\pi \in \Pi}$. Given $\mathbf{x}$, for each $\pi \in \Pi$, denote $\mathbf{x}_{|\pi} = \{x^{\pi'}\}_{\pi' \in \Pi_{|\pi}}$ and $\mathbf{x}^\pi = \mathbf{x}_{|\pi} \cup \{x^\pi\}$. Now we characterize an SSPE as a collection of initial-state stationary subgame perfect equilibria with respect to a specific value profile.

Proposition 1.

[15] This is known as a generalized Fubini Theorem. See Eraslan and McLennan (2013).

i) If $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ *is an SSPE, then it induces a value profile* $\mathbf{x}$ *and for all* $\pi \in \Pi$*, a partial strategy profile* (α^π, β^π) *is an initial-state stationary subgame perfect* $\mathbf{x}^\pi$*-equilibrium of the subgame starting with* π*.*

ii) If there exist a stationary strategy profile $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ *and a value profile* $\mathbf{x}$ *such that for all* $\pi \in \Pi$*,* (α^π, β^π) *is an initial-state stationary subgame perfect* $\mathbf{x}^\pi$*-equilibrium of the subgame starting with* π*, then* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ *is an SSPE.*

3.2 Cutoff Strategy Equilibria

A cutoff strategy profile $(\mathbf{x}, \mathbf{q})$ consists of a cutoff value profile $\mathbf{x} = \{\{x_i^\pi\}_{i \in N^\pi}\}_{\pi \in \Pi}$ and a coalition formation strategy profile $\mathbf{q} = \{\{q_i^\pi\}_{i \in N^\pi}\}_{\pi \in \Pi}$, where $x_i^\pi \in \mathbb{R}$ and $q_i^\pi \in \Delta(2^{N_i^\pi})$ for each $\pi \in \Pi$ and it specifies the behaviors of an active player $i \in N^\pi$ in any coalitional state π in the following way:

- player i proposes (S, y) with probability $q_i^\pi(S)$ such that

$$y_k = \begin{cases} x_k^\pi & \text{if } k \in S \\ 0 & \text{otherwise;} \end{cases}$$

- player i accepts any proposal (S, y) if and only if $y_i \geq x_i^\pi$.

Given $\mathbf{x}$, define an active player i's excess surplus of forming S in π:

$$e_i^\pi(S, \mathbf{x}) = x_i^{\pi(i,S)} - x_S^\pi.$$

Let $\mathcal{D}_i^\pi(\mathbf{x}) = \operatorname{argmax}_{S \subseteq N_i^\pi} e_i^\pi(S, \mathbf{x})$ be a *demand set* of player i in π and $m_i^\pi(\mathbf{x}) = \max_{S \subseteq N_i^\pi} e_i^\pi(S, \mathbf{x})$ be a *net proposal gain* of player i in π. Given a cutoff strategy profile $(\mathbf{x}, \mathbf{q})$, player i's *continuation payoff* in π is:

$$u_i^\pi(\mathbf{x}, \mathbf{q}) = p_i^\pi \sum_{S \subseteq N^\pi} q_i^\pi(S) e_i^\pi(S, \mathbf{x}) + \sum_{j \in N^\pi} p_j^\pi \sum_{S \subseteq N^\pi} q_j^\pi(S) \left[\mathbb{1}(i \in S) x_i^\pi + \mathbb{1}(i \notin S) x_i^{\pi(j,S)} \right]. \quad (3)$$

Now we show that, for any SSPE $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, there exists a cutoff strategy equilibrium $(\mathbf{x}, \mathbf{q})$ such that the SSPE $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ induces the value profile $\mathbf{x}$. Let $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be an SSPE. Due to Proposition 1, there exists a value profile $\mathbf{x}$ such that, for all $\pi \in \Pi$, (α^π, β^π) is an initial-state stationary subgame perfect $\mathbf{x}^\pi$-equilibrium of the subgame starting with π.

Proposition 2. *For an arbitrary SSPE, there exists a cutoff strategy equilibrium which yields the same expected payoff vector.*

Due to Proposition 2, when we are interested in players' equilibrium payoffs, without loss of generality, we can focus on a cutoff strategy equilibrium. The next proposition characterizes a cutoff strategy equilibrium in terms of a value profile and a coalition formation strategy profile.

Proposition 3. *Let* $0 < \delta < 1$*. A cutoff strategy profile* $(\mathbf{x}, \mathbf{q})$ *is an SSPE if and only if for all* $\pi \in \Pi$ *and* $i \in N^\pi$*,*

(Optimality) $q_i^\pi \in \Delta(\mathcal{D}_i^\pi(\mathbf{x}))$*; and*

(Consistency) $x_i^\pi = (1 - \delta) v([i]_\pi) + \delta u_i^\pi(\mathbf{x}, \mathbf{q})$.

4 Efficient Equilibria

A coalition $S \subseteq N$ is *efficient* if $v(S) = \bar{v}$. Let $\mathbf{E}$ be a set of *efficient coalitions*. When v is zero-normalized, essential, and monotone, $\mathbf{E}$ has the following properties:

(E1) $\{i\} \notin \mathbf{E}$ for all $i \in N$;

(E2) $S \in \mathbf{E}$ and $S \subset S'$ imply $S' \in \mathbf{E}$; and

(E3) $N \in \mathbf{E}$.

Let $K = \cap \mathbf{E}$ be a set of *essential players*. Let $\mathbf{E}^m = \{S \in \mathbf{E} \mid (\forall i \in S)\ S \setminus \{i\} \notin \mathbf{E}\}$ be a set of *minimal efficient coalitions* and $\mathbf{A} = \{A \subseteq N \setminus K \mid A \cup K \in \mathbf{E}^m\}$ be a set of *auxiliary coalitions*. Define $D = N \setminus (\cup \mathbf{E}^m)$ as a set of *dummy players*.

Remark. Note that the notion of an essential player is a generalization of a veto player in simple games. In a simple game, each coalition generates either a unit surplus or zero. A coalition that generates a unit surplus is called a winning coalition. We will discuss simple games as a special case in Section 4.3.

Lemma 1. *A player is essential if and only if the player has a positive marginal contribution to a grand-coalition, that is,* $k \in K \iff v(N \setminus \{k\}) < v(N)$.

Proof. By definition, $k \in K$ if and only if $S \in \mathbf{E} \implies k \in S$. First, for 'only-if' part, suppose $v(N \setminus \{k\}) < v(N)$ and take $S \in \mathbf{E}$. We need to show $k \in S$. Suppose not, that is, $S \subseteq N \setminus \{k\}$. Monotonicity implies $v(S) \leq v(N \setminus \{k\}) < v(N)$, which contradicts to $S \in \mathbf{E}$. Next, for 'if' part, suppose $S \in \mathbf{E} \implies k \in S$, or $k \notin S \implies S \notin \mathbf{E}$. Then we have $N \setminus \{k\} \notin \mathbf{E}$, which implies $v(N \setminus \{k\}) < v(N)$. □

Definition 1. Given $\Gamma = (N, v, p, \delta)$, a cutoff strategy profile $(\mathbf{x}, \mathbf{q})$ is *efficient* if

$$\sum_{i \in N} u_i(\mathbf{x}, \mathbf{q}) = \bar{v}.$$

Remark. Note that any proposal is always accepted in a cutoff strategy profile. Thus, a cutoff strategy profile $(\mathbf{x}, \mathbf{q})$ is efficient if and only if, for all $i \in N$ and $S \subseteq N_i$, $q_i(S) > 0$ implies $v(S) = \bar{v}$.

Before we state our main theorem, let us investigate the role of essential players. Consider a characteristic function form game (N, v) which has no essential player, that is, for all $i \in N$, $v(N \setminus \{i\}) = \bar{v}$. In such a game, there may exist an efficient equilibrium for all discount factors as in the following example.

Example 1 (Three-Player Majority Game)**.** Let $N = \{1, 2, 3\}$; and $v(S) = 1$ if $|S| \geq 2$, otherwise $v(S) = 0$. $\mathbf{E} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, N\}$ and $K = \emptyset$. Suppose $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. For a noncooperative bargaining game (N, v, p, δ) with any δ, there exists an equilibrium (x, q) such that for each $i \in N$ and $\{j, k\} = N \setminus \{i\}$

i) $x_i = \frac{1}{3}\delta$; and

ii) $q_i(\{i,j\}) = q_i(\{i,k\}) = \frac{1}{2}$.

In fact, in any equilibrium of (N, v, p, δ), a two-player coalition is always immediately formed, independently on p and δ.

Note that all the players are essential if and only if there exists a unique efficient coalition, which is the grand-coalition. In this case, in any efficient equilibrium, a grand-coalition is always immediately formed. This special efficient equilibrium is called a *grand-coalition equilibrium.* The extreme case of a single efficient coalition is a *unanimity game*, that is, $v(N) > 0$ and $v(S) = 0$ for all $S \subsetneq N$.

In our main theorem, we consider a underlying game with an essential player, that is, at least one player's marginal contribution to the grand-coalition is strictly positive. Then, it turns out that an efficient equilibrium is *generically impossible.* For any initial recognition probability, there is no efficient subgame perfect equilibrium for a sufficiently high discount factor, unless the underlying game is a unanimity game. Only a unanimity game has an efficient equilibrium for all discount factors.

Theorem 1.

i) *If (N, v) is a unanimity game, then for any p and δ, the bargaining game (N, v, p, δ) has an efficient stationary subgame perfect equilibrium .*

ii) *Suppose that (N, v) has an essential player and it is not a unanimity game. For any p there exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$, the bargaining game (N, v, p, δ) does not have any efficient stationary subgame perfect equilibrium.*

Remark. For a sufficiently low discount factor, an efficient equilibrium may exist. If players do not care about their future payoffs, then respondents will accept any offer. A proposer also does not care about the future bargaining power for a low discount factor, and hence she wants to form an efficient coalition immediately. As the discount factor increases, players consider their buyout options *more seriously* and form inefficient coalitions, and hence strategic delay may emerge in equilibria. In order for a bargaining game to have an efficient equilibrium with an arbitrary large discount factor, there are only two possible cases: either the underlying game is a unanimity game or it has no essential player.

Remark. As a transitional state, players may form an inefficient coalition, but an efficient coalition must be formed in a finite period. Thus, as the discount factor increase, tho different effects are intertwined. A higher discount factor promotes players' strategic alliance behavior, but it also reduces welfare loss. As the discount factor converges to 1, welfare loss eventually disappears while strategic delay remains.

The following example illustrates why non-unanimity games have no efficient equilibrium: if we assume an efficient strategy profile is an equilibrium, then there exists a player who can be better off by forming an inefficient coalition for a sufficiently high discount factor. The actual equilibrium is investigated in Section 7.

Example 2. $N = \{1,2,3\}$; $v(N) = 1$, $v(\{1,2\}) = v(\{1,3\}) = a$, $v(S) = 0$ otherwise. Suppose $0 \le a < 1$ and $p_i = \frac{1}{3}$ for all $i \in N$. Suppose $(\mathbf{x},\mathbf{q})$ be an efficient equilibrium. Then a grand-coalition is always immediately formed and each player is always chosen by any proposer. Hence each player i's expected payoff is

$$u_i(\mathbf{x},\mathbf{q}) = p_i(1 - x_N) + x_i. \tag{4}$$

Since it is efficient, it must be $u_N(\mathbf{x},\mathbf{q}) = 1$. Due to consistency condition, that is $x_i = \delta u_i(\mathbf{x},\mathbf{q})$ for all i, (4) yields $(1-\delta)x_i = \delta p_i(1-\delta)$, or $x_i = \delta p_i$. Then player 1's excess surpluses are

- $e_1(N,\mathbf{x}) = 1 - x_N = 1 - \delta$; and
- $e_1(\{1,2\},\mathbf{x}) = e_1(\{1,3\},\mathbf{x}) = \underbrace{a + \dfrac{2}{3}\delta(1-a)}_{x_1^{(1,\{1,2\})}} - \underbrace{\dfrac{2}{3}\delta}_{x_1+x_2} = a - \frac{2}{3}a\delta.$

Note that $x_1^{(1,\{1,2\})} = x_1^{(1,\{1,3\})} = a + \frac{2}{3}\delta(1-a)$, because player 1 derives a by forming $\{1,2\}$ or $\{1,3\}$ and she additionally expects $\frac{2}{3}(1-a)$ from the two-player game in the next period. Let $\bar{\delta} = \frac{1-a}{1-\frac{2}{3}a}$. Then $e_1(\{1,2\},\mathbf{x}) > e_1(N,\mathbf{x})$ for all $\delta > \bar{\delta}$. If $a > 0$, then $\bar{\delta} < 1$, and hence for all $\delta > \bar{\delta}$ froming a grand-coalition is not optimal for player 1, which violates $(\mathbf{x},\mathbf{q})$ is an equilibrium. Now suppose that $a = 0$, that is (N,v) is a unanimity game. Then $\bar{\delta} = 1$ and there is no $\delta < 1$ which contradicts to the assumptions of an efficient equilibrium. Therefore, the efficient strategy profile $(\mathbf{x},\mathbf{q})$ is indeed an equilibrium for all δ.

We will prove the main theorem in the following two subsections. Subsection 4.1 considers a game in which all the players are essential. In such games, a grand-coalition is the unique efficient coalition and hence any efficient equilibrium is a grand-coalition equilibrium. Proposition 4 constructs a grand-coalition equilibrium for unanimity games. For unanimity games, we will see that players' equilibrium payoffs are proportional to their recognition probabilities. Then, players cannot improve their bargaining power by forming a strict subcoalition, because it just induces another unanimity game. Thus, all the players always immediately form a grand-coalition independently on discount factors. Conversely, in Proposition 5, we show that a grand-coalition equilibrium is impossible for other than a unanimity game, if the discount factor is sufficiently high.

The novel (and also hard) part of our theorem is dealing with a game with multiple efficient coalitions, while both Seidmann and Winter (1998) and Okada (2000) restricted on the cases with a single efficient coalition.[16] Subsection 4.2 extends the inefficiency result into the games with multiple efficient coalitions. Proposition 6 shows that, for any non-unanimity game with an essential player, any efficient strategy profile cannot be an equilibrium for sufficiently high discount factor. The proof consists of three cases. In the first case, $K \notin \mathbf{E}$,[17] non-essential players form a coalition among themselves to be a new essential player in the subsequent bargaining game. In the second case, $K \in \mathbf{E}$

[16] Seidmann and Winter (1998) assumes *semi-strict superadditivity*, which is the weakest assumption to imply that a grand-coalition is the unique efficient coalition. Okada (2000) assumes that one and only one profitable coalition can be formed.

[17] The set of essential players is not an efficient coalition, and hence essential players need at least one non-essential player's cooperation to for an efficient coalition.

and there exists $k' \in K$ such that $v(N \setminus \{k'\}) > 0$, another essential player $k \in K$ can be better off by forming an inefficient coalition excluding k'. In the last case, $K \in \mathbf{E}$ and $v(N \setminus \{k\}) = 0$ for all for $k \in K$, each essential player forms a coalition with dummy players rather than immediately forming an efficient coalition.

In Subsection 4.3, we rephrase our results for simple games.

4.1 Grand-Coalition Equilibria

In this section, we show that a grand-coalition equilibrium does not exist in general if the players have buyout options. That is, players may form a transitional subcoalition and a grand-coalition, which is the unique efficient coalition, is delayed to form. The only case in which a grand-coalition equilibrium exists is a unanimity game, that is, all the subcoalitions except for the grand coalition produce nothing.

First, we characterizes the payoff vector in grand-coalition equilibria. Though we assume zero-normalization on initial characteristic functions, but in non-initial states, the induced characteristic functions may not be zero-normalized. The following lemma does not rely on zero-normalization.

Lemma 2. *Suppose* $(\mathbf{x}, \mathbf{q})$ *is a grand-coalition equilibrium of* (N, v, p, δ)*. For each* $i \in N$*,*

1) $u_i(\mathbf{x}, \mathbf{q}) = v_i + p_i(\bar{v} - v_N)$*; and*

2) $x_i = v_i + \delta p_i(\bar{v} - v_N)$*.*

Proof. Since $(\mathbf{x}, \mathbf{q})$ is efficient, it must be $u_N(\mathbf{x}, \mathbf{q}) = \bar{v}$ and $x_N = \delta\bar{v}$. Take $i \in N$. i always forms a grand-coalition and i is always included by other players' proposal as well. Thus, i's expected equilibrium payoff is

$$u_i(\mathbf{x}, \mathbf{q}) = p_i(\bar{v} - x_N) + x_i = p_i(1 - \delta)\bar{v} + x_i. \tag{5}$$

Consistency condition in Proposition 3 requires

$$x_i = (1 - \delta)v_i + \delta u_i(\mathbf{x}, \mathbf{q}) \tag{6}$$

and hence (5) yields the first result. Plugging the first result into (6), the second result follows. □

In unanimity games, each player always immediately forms a grand-coalition and the expected payoff is proportional to her recognition probability. More precisely, for each state $\pi \in \Pi$ and each active player $i \in N^\pi$, her value in the state is $x_i^\pi = \delta\bar{v}p_i^\pi$ and her expected payoff in the initial state is $p_i\bar{v}$.

Proposition 4. *Let* (N, v) *be a unanimity game. For any* (N, v, p, δ)*, there exists a grand-coalition equilibrium and the equilibrium payoff vector is unique.*

Proof. For any two-player game, it is clearly true. As an induction hypothesis, suppose the statement is true for any less-than-n-player game. Now consider an n-player game (N, v, p, δ). Let $\delta\bar{v}\mathbf{p}$ be a value profile with $x_i^\pi = \delta\bar{v}p_i^\pi$ for each $\pi \in \Pi$ and $i \in N^\pi$. Let $\bar{\mathbf{q}}$ be a formation strategy profile with $\bar{q}_i^\pi(N^\pi) = 1$ for each $\pi \in \Pi$ and $i \in N^\pi$.

Step 1: $(\delta\bar{v}\mathbf{p}, \bar{\mathbf{q}})$ is an equilibrium.

For any non-initial state π, the strategy profile is an equilibrium in the subgame due to the induction hypothesis and Lemma 2. We need to verify $(\delta\bar{v}p, \bar{q})$ is an equilibrium in the initial state. First, consistency condition in Proposition 3 is hold by Lemma 2. Second, for each $i \in N$ and $S \subsetneq N_i$,

$$e_i(S, \delta\bar{v}\mathbf{p}) = x_i^{(i,S)} - x_S = \delta\bar{v}p^{(i,S)} - \delta\bar{v}p_S = 0.$$

However, $e_i(N, \delta\bar{v}\mathbf{p}) = \bar{v} - \delta\bar{v}p_N = (1-\delta)\bar{v} > 0$, and hence forming a grand-coalition satisfies optimality condition.

Step 2: The equilibrium payoff vector is unique.

Let $(\mathbf{x}, \mathbf{q})$ be an equilibrium. Again we need to check the uniqueness only for the initial state. Suppose (x, q) is an equilibrium in the initial state. If $q = \bar{q}$, then Lemma 2 implies $x = \delta\bar{v}p$, which yileds the same equilibrium payoff vector. Suppose that there exists $i \in N$ such that $q_i(S) > 0$ with $S \subsetneq N$. Then this equilibrium is inefficient and hence it must be $x_N < \delta\bar{v}$. Also i's optimality condition requires that $x_i^{(i,S)} - x_S \geq \bar{v} - x_N$. The induction hypothesis implies that $\delta\bar{v}p_S - x_S \geq \bar{v} - x_N$. Putting $x_N < \delta\bar{v}$, we have

$$x_S < \delta\bar{v}p_S - (1-\delta)\bar{v}. \tag{7}$$

On the other hand, letting $Q_j = \sum_{k\in N} p_k \sum_{S\subseteq N} q_k(S)\mathbb{1}(j \in S)$, for each $j \in S$, we have

$$\begin{aligned} u_j(\mathbf{x}, \mathbf{q}) &\geq p_j(\bar{v} - x_N) + Q_j x_j + (1-Q_j)\delta p_j \bar{v} \\ &> p_j\bar{v} + Q_j(x_j - \delta p_j\bar{v}). \end{aligned} \tag{8}$$

Since $\delta u_j(\mathbf{x}, \mathbf{q}) = x_j$, rearranging (8), we have $x_j > \delta p_j \bar{v}$. Since this inequality holds for any $j \in S$, summing this over S, it follows $x_S > \delta\bar{v}p_S$. However, this contradicts to (7). Therefore, for all $i \in N$, it must be $q_i(N) = 1$ in any equilibrium. □

Now we prove impossibility of grand-coalition equilibria for non-unanimity games.

Proposition 5. *Suppose (N, v) is not a unanimity game. For any p, there exists $\bar{\delta} < 1$ such that, for all $\delta > \bar{\delta}$, a bargaining game (N, v, p, δ) has no grand-coalition equilibrium.*

Proof. Since (N, v) is not a unanimity game, there exists $S \subsetneq N$ such that $v(S) > 0$. By superadditivity, for all $k \in N \setminus S$, we have $v(N \setminus \{k\}) > 0$. Suppose $(\mathbf{x}, \mathbf{q})$ is an efficient equilibrium. Take any $i \in S$. Player i's optimality condition requires $e_i(N, \mathbf{x}) \geq e_i(N \setminus \{k\}, \mathbf{x})$ or

$$x_i^{(i,N)} - x_N \geq x_i^{(i,N\setminus\{k\})} - x_N + x_k. \tag{9}$$

First, Lemma 2 implies that $x_i^{(i,N)} = \bar{v}$ and $x_k = \delta p_k \bar{v}$. Secondly, i's $(N \setminus \{k\})$-formation induces a two-player game, and hence Lemma 2 yields $x_i^{(i,N\setminus\{k\})} = v(N \setminus \{k\}) + \delta(1-p_k)\left(\bar{v} - v(N \setminus \{k\})\right)$. Plugging x_k, $x_i^{(i,N)}$, and $x_i^{(i,N\setminus\{k\})}$ into (9), we have

$$\bar{v} \geq v(N \setminus \{k\}) + \delta(1-p_k)\left(\bar{v} - v(N \setminus \{k\})\right) + \delta p_k \bar{v}.$$

Rearranging the terms, it follows

$$(1-\delta)\bar{v} \leq (1-\delta(1-p_k))v(N \setminus \{k\}). \tag{10}$$

As $\delta \to 1$, the left-hand side of (10) converges to zero; while the right-hand side is strictly positive uniformly on δ. More precisely, letting $\bar{\delta} = \frac{v(N\setminus\{k\})}{(1-p_k)v(N\setminus\{k\})+p_k\bar{v}}$, for all $\delta > \bar{\delta}$, forming a grand-coalition is not optimal for i, and hence $(\mathbf{x}, \mathbf{q})$ cannot be an equilibrium. □

The following result is a direct consequence of Proposition 4 and Proposition 5.

Corollary 1. *A bargaining game (N, v, p, δ) has a grand-coalition equilibrium for all discount factors if and only if (N, v) is a unanimity game.*

4.2 Efficient Equilibria with Multiple Efficient Coalitions

In previous subsection, we focused on a grand-coalition equilibrium, which is a special class of efficient equilibria. If the grand-coalition is a unique efficient coalition, then an efficient equilibrium must be a grand-coalition equilibrium. In this subsection, we consider underlying games with multiple efficient coalitions. That is, the maximum surplus can be generated by strict subcoalitions other than a grand-coalition. Hence strict subcoalition may be formed as a terminal state in efficient equilibria. Proposition 6 generalizes the inefficiency result into games with multiple efficient coalitions. That is, if there exists an essential player, then an efficient equilibrium is impossible for a sufficiently high discount factor.

Proposition 6. *Suppose (N, v) has an essential player and it is not a unanimity game. There exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$ any efficient strategy profile $(\mathbf{x}, \mathbf{q})$ cannot be an equilibrium.*

Before we prove this proposition, we need three lemmas, which show lower-bounds of players' payoffs under the assumption of an efficient equilibrium. Lemma 3 shows that players get at least some positive part of their marginal contribution to a grand-coalition in addition to their stand-alone value, when the grand-coalition is the unique efficient coalition.

Lemma 3. *Suppose (N, v) has a unique efficient coalition. If $(\mathbf{x}, \mathbf{q})$ is an equilibrium of (N, v, p, δ), then for all $i \in N$*

$$u_i(\mathbf{x}, \mathbf{q}) \geq v(\{i\}) + \delta^{|N|-2} p_i \left(\bar{v} - v(N \setminus \{i\})\right).$$

Proof. If $|N| = 2$, then there exists a unique equilibrium which is efficient, and hence Lemma 2 implies the result as an equality. As induction hypothesis, suppose the result holds for any less-than-n-player game. Consider (N, v) with a unique efficient coalition and $|N| = n$. For all $j \in N$ and $S \subseteq N_j$ with $|S| \geq 2$, the subsequent game $(N^{(j,S)}, v^{(j,S)})$ has a unique efficient coalition and the induction hypothesis yields, for all $i \in N \setminus S$,

$$\begin{aligned} u_i^{(j,S)}(\mathbf{x}, \mathbf{q}) &\geq v^{(j,S)}(\{i\}) + \delta^{n-3} p_i^{(j,S)}(\bar{v} - v^{(j,S)}(N \setminus \{i\})) \\ &= v(\{i\}) + \delta^{n-3} p_i(\bar{v} - v(N \setminus \{i\})) \end{aligned} \tag{11}$$

Using consistency condition and (11), now we calculate the lower bound of $u_i(\mathbf{x}, \mathbf{q})$.

$$\begin{aligned} u_i(\mathbf{x}, \mathbf{q}) &\geq p_i \cdot 0 + \sum_{j \in N} p_j \sum_{S \subseteq N} q_j(S) \left(\mathbb{1}(i \in S) x_i + \mathbb{1}(i \notin S) x_i^{(j,S)} \right) \\ &\geq Q_i \left((1-\delta) v(\{i\}) + \delta u_i(\mathbf{x}, \mathbf{q}) \right) \\ &\quad + (1 - Q_i) \left((1-\delta) v(\{i\}) + \delta \left(v(\{i\}) + \delta^{n-3} p_i^{\pi} \left(\bar{v} - v(N \setminus \{i\}) \right) \right) \right) \\ &= \delta Q_i u_i(\mathbf{x}, \mathbf{q}) + (1 - \delta Q_i) v(\{i\}) + (1 - Q_i) \delta^{n-2} (\bar{v} - v(N \setminus \{i\})), \end{aligned}$$

where $Q_i = \sum_{j \in N} p_j \sum_{S \subseteq N} q_j(S) \mathbb{1}(i \in S)$. Rearranging the terms, the inequality yields

$$\begin{aligned} u_i(\mathbf{x}, \mathbf{q}) &\geq v(\{i\}) + \frac{1 - Q_i}{1 - \delta Q_i} \delta^{n-2} (\bar{v} - v(N \setminus \{i\})) \\ &\geq v(\{i\}) + \delta^{n-2} (\bar{v} - v(N \setminus \{i\})), \end{aligned}$$

as desired. □

Lemma 4 provides a lower bound for each essential player under assuming an efficient equilibrium.

Lemma 4. *Suppose $(\mathbf{x}, \mathbf{q})$ is an efficient equilibrium. For any $k \in K$,*

$$x_k = \frac{\delta}{1-\delta} p_k (\bar{v} - x_K) = \frac{p_k}{p_K} x_K \geq \delta p_k \bar{v}.$$

Proof. Since any essential player must be offered no matter who makes a proposal, for each $k \in K$, we have $u_k(\mathbf{x}, \mathbf{q}) = p_k(\bar{v} - x_K) + x_k$, or equivalently due to consistency

$$(1-\delta) x_k = \delta p_k (\bar{v} - x_K), \tag{12}$$

which implies the first equality. Summing (12) over K, we have $(1-\delta)x_K = \delta p_K (\bar{v} - x_K)$. Plugging this into (12) again, we have the second equality part. Since $u_N(\mathbf{x}, \mathbf{q}) \leq \bar{v}$, we have $x_K \leq x_N \leq \delta \bar{v}$ and hence $(1-\delta)x_k \geq \delta p_k (1-\delta) \bar{v}$, which implies the inequality part. □

The Lemma 5 shows that all non-essential players get almost nothing as the discount factor closes to 1, under any efficient equilibrium. This lemma is a generalized version of Winter (1996); he showed that non-veto players get nothing in equilibria for the class of simple games with a uniform recognition probability and no dummy player. However, we will use this lemma to show a contradiction in an efficient equilibrium. That is, when players have buyout options, even non-essential players get strictly positive payoff uniformly on discount factors, and hence efficient equilibria is impossible.

Lemma 5. *Suppose $(\mathbf{x}, \mathbf{q})$ is an efficient equilibrium. If $K \neq \emptyset$, then x_K converges to $\bar{v}$ and $x_{N \setminus K}$ to 0 as $\delta \to 1$.*

Proof. Suppose $(\mathbf{x}, \mathbf{q})$ is an efficient equilibrium. Each non-essential player $j \in N \setminus K$ must form a coalition $K \cup \{j\}$. Thus player j's payoff is $u_j(\mathbf{x}, \mathbf{q}) = p_j(\bar{v} - x_K - x_j) + p_j x_j$. Multiplying δ to both sides and rearranging terms, we have $x_j = \delta p_j (\bar{v} - x_K)$. Summing this over $N \setminus K$, we have

$x_{N\setminus K} = \delta p_{N\setminus K}(\bar{v} - x_K)$. For essential players, by Lemma 4, we have $x_K = \frac{\delta}{1-\delta} p_K(\bar{v} - x_K)$. Since $(\mathbf{x}, \mathbf{q})$ is efficient, $u_N(\mathbf{x}, \mathbf{q}) = \bar{v}$ and hence $x_N = \delta u_N(\mathbf{x}, \mathbf{q}) = \delta \bar{v}$. Altogether, we have

$$\bar{v} = \frac{x_N}{\delta} = \frac{x_K + x_{N\setminus K}}{\delta} = p_K\left(\frac{\bar{v} - x_K}{1-\delta}\right) + p_{N\setminus K}(\bar{v} - x_K).$$

Since $\bar{v}$ is bounded and $p_K > 0$, $\frac{\bar{v}-x_K}{1-\delta}$ must be bounded and the result follows. □

Now we are ready to prove Proposition 6. There are three possible cases. First, the essential players can be an efficient coalition by themselves, that is, $v(K) = v(N)$. Otherwise, the essential players need some other players' cooperation to form an efficient coalition, that is, $v(K) < v(N)$. The second case is, in addition to $v(K) < v(N)$, when there exists $k' \in K$ such that $v(N \setminus \{k'\}) > 0$. In the last case, we consider the case that $v(K) < v(N)$ and $v(N \setminus \{k\}) = 0$ for all $k \in K$. In each case, we observe different strategic reasons which yield inefficiency.

Case 1: $K \notin \mathbf{E}$

In this case, K needs non-essential players' cooperation to form an efficient coalition. Then non-essential players form a coalition to become a new essential player in the subsequent game, rather than directly forming an efficient coalition.

Proof of Proposition 6 (Case 1):

If $K \notin \mathbf{E}$, then $v(K) < \bar{v}$. Hence, there exists $A \neq \emptyset$ such that $A \in \mathbf{A}$. Take $i \in A$ and suppose $(\mathbf{x}, \mathbf{q})$ is an efficient equilibrium. Player i's optimality condition requires that $e_i(K \cup A, \mathbf{x}) \geq e_i(N \setminus K, \mathbf{x})$, which is

$$\bar{v} - x_K - x_A \geq x_i^{(i, N\setminus K)} - x_N + x_K. \tag{13}$$

After i's $N \setminus K$-formation, no dummy player exists any more and there is only one efficient coalition in the subsequent state. Thus Lemma 3 implies that

$$\begin{aligned} \bar{v} - x_K - x_A &\geq v(N \setminus K) + \delta^{|K|-2}(1 - p_K)(\bar{v} - v(K)) - x_N + x_K \\ (1+\delta)\bar{v} - 2x_K - x_A &\geq \delta^{|K|-2}(1 - p_K)(\bar{v} - v(K)), \end{aligned} \tag{14}$$

where the second inequality comes from $x_N = \delta \bar{v}$ in an efficient equilibrium. Due to Lemma 5, as $\delta \to 1$, the left-hand side of (14) converges to 0; while the right-hand side is strictly positive, which yields a contradiction. □

Case 2: $K \in \mathbf{E}$ and there exists $k' \in K$ such that $v(N \setminus \{k'\}) > 0$

Note that if $K \in \mathbf{E}$, then there are at least two distinct essential players $k, k' \in K$; otherwise a singleton is an efficient coalition, which contradicts to zero-normalization. If $D = \emptyset$, then N is the unique efficient coalition, which we discussed in the previous subsection. Hence we assume that $D \neq \emptyset$. Also if $K \in \mathbf{E}$, then K is the unique minimal efficient coalition and $D = N \setminus K$. We will show that at least one of essential players can be better off by excluding the other essential player if all the players are supposed to play efficient strategies.

Proof of Proposition 6 (Case 2): Take $k \in K$ such that $k \neq k'$. Let $(\mathbf{x}, \mathbf{q})$ be an efficient equilibrium. Player k's optimality condition implies that $e_k(K, \mathbf{x}) \geq e_k(N \setminus \{k'\}, \mathbf{x})$, that is,

$$\bar{v} - x_K \geq x_k^{(k, N \setminus \{k'\})} - x_{N \setminus \{k'\}}. \tag{15}$$

Since k's $N \setminus \{k'\}$-formation yields a two-player game, k's value in the subsequent state is

$$x_k^{(k, N \setminus \{k'\})} = v(N \setminus \{k'\}) + \delta(1 - p_{k'})(\bar{v} - v(N \setminus \{k'\}). \tag{16}$$

Plugging (16) into (15), we have

$$\begin{aligned} (1 - \delta(1 - p_{k'}))\bar{v} - x_K &\geq (1 - \delta(1 - p_{k'}))v(N \setminus \{k'\}) - x_N + x_{k'} \\ (1 + \delta p_{k'})\bar{v} - x_K &\geq (1 - \delta(1 - p_{k'}))v(N \setminus \{k'\}) + x_{k'} \\ \bar{v} - x_K &\geq (1 - \delta(1 - p_{k'}))v(N \setminus \{k'\}), \end{aligned}$$

where the second line comes from efficiency $x_N = \delta \bar{v}$; and the third line is due to Lemma 4. However, by Lemma 5, the left-hand side converges to 0 as $\delta \to 1$; while the right-hand side is strictly positive. □

Case 3: $K \in \mathbf{E}$ **and** $\forall k \in K \; v(N \setminus \{k\}) = 0$

If $v(N \setminus \{k'\}) = 0$ for all $k' \in K$, then it may not be profitable for any essential player k to exclude other essential player k'. In this case, we show that it is profitable for $k \in K$ to form a coalition with all the dummy players rather than forming an efficient coalition.

Proof of Proposition 6 (Case 3):

First, we find an strict inequality which holds independently on δ. Since $|K| \geq 2$, we have $p_K > p_k$ and hence $(1 - p_K)(p_K - p_k) > 0$. Rearranging terms and using $p_D + p_K = 1$, it follows that

$$1 - p_D - p_k < 1 - \frac{p_k}{p_K}. \tag{17}$$

Now suppose $(\mathbf{x}, \mathbf{q})$ is an efficient equilibrium. Take any $k \in K$. Player k's optimality condition implies that $e_k(K, \mathbf{x}) \geq e_k(D \cup \{k\}, \mathbf{x})$, that is,

$$\bar{v} - x_K \geq x_k^{(k, D \cup \{k\})} - x_D - x_k. \tag{18}$$

After k's $D \cup \{k\}$-formation, no dummy player exists any more and there is only one efficient coalition in the subsequent state. Thus Lemma 3 implies that

$$\bar{v} - x_K \geq v(D \cup \{k\}) + \delta^{|K|-1}(p_D + p_k)(\bar{v} - v(K \setminus \{k\}) - x_D - x_k. \tag{19}$$

Since $v(K \setminus \{k\}) = 0$ and $x_k = \frac{p_k}{p_K} x_K$ due to Lemma 4, (19) yields

$$\left(1 - \delta^{|K|-1}(p_D + p_k)\right)\bar{v} + x_D \geq \left(1 - \frac{p_k}{p_K}\right)x_K. \tag{20}$$

Due to Lemma 5, as $\delta \to 1$, (20) requires that $1 - p_D - p_k \geq 1 - \frac{p_k}{p_K}$, which contradicts to (17) □

The following corollary is a direct consequence of the proof of Proposition 6. The corollary states the inefficiency result with respect to the final coalition; while Corollary 2 states about delay in an equilibrium.

Corollary 2. *Suppose (N, v) has an essential player and it is not a unanimity game. There exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$ a non-minimal efficient coalition occurs in the terminal state with positive probability in any equilibrium.*

4.3 Simple Games

We apply the result to simple games. Given a set of players N, a class of subsets $\mathbf{W} \subset 2^N$ is a set of *winning coalitions* if

i) $\{i\} \notin \mathbf{W}$ for all $i \in N$; and

ii) $S \in \mathbf{W}$ and $S \subset S'$ imply $S' \in \mathbf{W}$.

A characteristic function form game (N, v) is *simple* if $v(S) = 1$ for all $S \in \mathbf{W}$ and $v(S) = 0$ otherwise.[18] Let $\mathbf{W}^m = \{S \in \mathbf{W} \mid (\forall i \in S)\ S \setminus \{i\} \notin \mathbf{W}\}$ be a set of *minimal winning coalitions*, $V = \cap \mathbf{W}$ a set of *veto players*, and $D = N \setminus (\cup \mathbf{W}^m)$ a set of *dummy players*. We also define a set of *auxiliary coalitions* $\mathbf{A} = \{A \subseteq N \setminus V \mid A \cup V \in \mathbf{W}^m\}$. First, we re-state our main inefficiency results to simple games with transferable recognition probabilities.

Corollary 3. *Let (N, v) be a simple game with a veto player. A bargaining game (N, v, p, δ) has an efficient equilibrium for all discount factors if and only if it is unanimous.*

Corollary 4. *Let (N, v) be a simple game with a veto player and multiple winning coalitions. In an equilibrium of (N, v, p, δ), there exist $\bar{\delta} < 1$ such that, for all $\delta > \bar{\delta}$, a non-minimal winning coalition forms with positive probability in the terminal state. Furthermore, the equilibrium expected payoff vector is not in the core.*

5 Non-transferability of Recognition Probabilities

We have assumed that the initial recognition probabilities are transferable so that when they trade their resources and rights, they also trade their chances to be a proposer as well. With transferable recognition probabilities, our result verified that efficient equilibria do not exist if the underlying game is a non-unanimity game with an essential player. The inefficiency comes from players' strategic decision on coalition formation. To be concrete, when each player chooses a coalition, they consider three different effects to choose a subcoalition: 1) combining resources, 2) changing coalition structure, and 3) increasing the chance to be a proposer. The first one generates a surplus in the current period, and the other two effects work for the future bargaining power and the expected payoff.

[18] Simple games are introduced by von Neumann and Morgenstern (1944). See Shapley (1962) for mathematical properties of simple games.

Now we eliminate the last effect of coalition formation and investigate the role of recognition probabilities. Assume that players' recognition probabilities are their innate right which cannot be traded. That is, each proposer inherits other respondents' resources but not their chances to be a proposer. With non-transferable recognition probabilities, instead of (1), we define the recognition probabilities in any state π in the following way:

$$p_i^\pi = \begin{cases} \dfrac{p_i}{\sum_{k \in N^\pi} p_k} & \text{if } i \in N^\pi \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Remark. In Gul (1989), every meetings are always randomly selected equally likely in any coalitional state. That is, the initial recognition probability is uniform and players *cannot* trade their recognition probabilities. The uniformity and non-transferability of recognition probabilities are crucial to implement the Shapley value. He also considered an alternative environment, namely a partnership game, to observe the implementation is not robust. One can view his partnership game environment as a special case in which the initial recognition probability is uniform and players can trade their recognition probabilities.

If we do not allow players to trade their recognition probabilities, in a broader class of characteristic function form games, an efficient equilibrium exists. We characterize a sufficient condition for efficient equilibria. In this section, we assume that a grand-coalition is a unique efficient coalition. Thus, in any efficient equilibrium, a grand-coalition must be always immediately formed. For any $\pi \in \Pi$ and $S \subseteq N^\pi$, let $[S]_\pi = \sum_{j \in S} [j]_\pi$. Also define $v^\pi(S) = v([S]_\pi) - \sum_{j \in S} v([j]_\pi)$, which represents additional surplus by S-formation in π. Note that $v^\pi(N^\pi) = \bar{v} - \sum_{j \in N^\pi} v([j]_\pi)$. Define a value profile $\bar{\mathbf{x}}$ such that for all $\pi \in \Pi$ and $i \in N^\pi$,

$$\bar{x}_i^\pi = v([i]_\pi) + \delta p_i^\pi v^\pi(N^\pi). \tag{22}$$

Lemma 6. *For all $\pi \in \Pi$, $i \in N^\pi$, and $S \subsetneq N_i$ such that $|S| \geq 2$,*

$$e_i^\pi(N^\pi, \bar{\mathbf{x}}) \geq e_i^\pi(S, \bar{\mathbf{x}}) \iff v^\pi(N^\pi) \geq \rho_i^\pi(S;\delta) v^\pi(S),$$

where $\rho_i^\pi(S;\delta) = \dfrac{1 - p_S^\pi + p_i^\pi - \delta p_i^\pi}{(1 - \delta + \delta p_S^\pi)(1 - p_S^\pi + p_i^\pi) - \delta p_i^\pi}$.

Proof. Fix $\pi \in \Pi$, $i \in N^\pi$, and $S \subsetneq N_i$ with $|S| \geq 2$. Using (22), calculate $e_i^\pi(N^\pi, \bar{\mathbf{x}})$ and $e_i^\pi(S, \bar{\mathbf{x}})$:

$$\begin{aligned} e_i^\pi(N^\pi, \bar{\mathbf{x}}) &= \bar{v} - \bar{x}_{N^\pi}^\pi \\ &= \bar{v} - \sum_{j \in N^\pi} v([j]_\pi) - \delta v^\pi(N^\pi) \\ &= (1-\delta) v^\pi(N^\pi), \end{aligned} \tag{23}$$

and

$$
\begin{aligned}
e_i^\pi(S,\bar{\mathbf{x}}) &= x_i^{\pi(i,S)} - x_S^\pi \\
&= v([S]_\pi) + \delta \frac{p_i^\pi}{1-p_S^\pi+p_i^\pi}\left(\bar{v} - \sum_{j\in N^\pi} v([j]_\pi) + \sum_{j\in S} v([j]_\pi) - v([S]_\pi)\right) \\
&\quad - \left(\sum_{j\in S} v([j]_\pi) + \delta p_S^\pi v^\pi(N^\pi)\right) \\
&= \delta\left(\frac{p_i}{1-p_S+p_i} - p_S\right) v^\pi(N^\pi) + \left(1-\delta\frac{p_i}{1-p_S+p_i}\right) v^\pi(S). \qquad (24)
\end{aligned}
$$

From (23) and (24), we have the desired result. □

Proposition 7. *Suppose that players cannot transfer their recognition probabilities. There exists an equilibrium in which a grand-coalition is always immediately formed in all states for all discount factors, if and only if, for all $\pi \in \Pi$, $i \in N^\pi$, and $S \subsetneq N_i$ such that $|S| \geq 2$,*

$$v^\pi(N^\pi) \geq \left(\frac{1}{p_S^\pi - p_i^\pi}\right) v^\pi(S). \qquad (25)$$

Proof. Suppose $\bar{\mathbf{q}}$ constructs an equilibrium. Due to Lemma 2, the corresponding value profile is uniquely determined as defined in (22). By Lemma 6, optimality condition implies for all δ,

$$v^\pi(N^\pi) \geq \rho_i^\pi(S;\delta) v^\pi(S).$$

Observe that $\frac{1}{p_S^\pi - p_i^\pi} \geq \rho_i^\pi(S;\delta)$ for all $\delta \leq 1$, and $\rho_i^\pi(S;\delta) \to \frac{1}{p_S^\pi - p_i^\pi}$ as $\delta \to 1$. This completes the necessary condition. Now suppose that for all $\pi \in \Pi$, $i \in N^\pi$, and $S \subsetneq N_i$ such that $|S| \geq 2$, the inequality (25) holds. Then it can be easily shown that a strategy profile $(\bar{\mathbf{x}}, \bar{\mathbf{q}})$ satisfies optimality condition and consistency condition for all δ. □

The following lemma is the special case in which the initial recognition probability is uniform.

Corollary 5. *Suppose that all the active players are selected as a proposer equally likely in any state. There exists an equilibrium in which a grand-coalition is always immediately formed in all states for all discount factors, if and only if, for all $\pi \in \Pi$ and $S \subsetneq N$ such that $|S| \geq 2$,*

$$\frac{v^\pi(N^\pi)}{|N^\pi|} \geq \frac{v^\pi(S)}{|S|-1}. \qquad (26)$$

Proposition 7 and its corollary provide sufficient conditions for a grand-coalition equilibrium. A grand-coalition equilibrium requires that all the players form a grand-coalition only in the initial state. As long as players are supposed to form a grand-coalition immediately, their strategies in proper subgames do not effect on efficiency. In other words, players threat each other by some inefficient strategies along off-equilibrium paths to support an efficient equilibrium. In Proposition 7 and its corollary, we investigate a stronger notion of efficiency; it requires efficiency even in off-equilibrium paths.[19] Thus the condition in Proposition 7 is a sufficient condition for a grand-coalition equilibrium to exist. Thus, with non-transferable recognition probabilities, if larger coalitions produce more, then the grand-coalition is always immediately formed.

The condition may not be a necessary condition. Characterizing a necessary condition for a grand-coalition equilibrium seems complicated in general with non-transferable recognition probabilities. However, at least for three-player games, the condition in Proposition 7 (and its corollary) is also a necessary condition for a grand-coalition equilibrium. For any non-initial non-terminal state in three-player games, there are only two active players and hence they form a grand-coalition as long as it is the unique efficient coalition.

Example 3. $N = \{1, 2, 3\}$; and $v(S) = |S|$ if $|S| \geq 2$, and $v(\{i\}) = 0$ for all $i \in N$. Without buyout options, players always immediately form a grand-coalition for all $\delta < 1$. Now suppose players have buyout options. With non-transferable recognition probabilities, an efficient equilibrium exists if and only if $\delta < \frac{2}{3}$. If recognition probabilities are transferable, it exists if and only if $\delta < \frac{3}{5}$.

Remark. Proposition 7 implies that efficiency can be improved by banning players from trading recognition probabilities. If they trade their recognition probabilities, then they can expected stronger bargaining power in the subsequent game by forming a transitional subcoalition rather than immediately forming an efficient coalition. Thus, the effect of transferability of recognition probabilities on efficiency may be negative. However, this effect can be positive when cooperation restrictions are imposed. As discussed in Lee (2013), efficiency can be improved by allowing for players to trade their recognition probabilities in network-restricted unanimity games. This is because, with cooperation restrictions, externalities on recognition probabilities are involved if they cannot trade recognition probabilities. That is, when a player forms a subcoalition, the recognition probabilities of the players out of the coalition also increase. Such externalities make players defer forming a coalition and behave inefficiently.

6 Application I : Simple Games

6.1 Three-player Simple Games

As an application, we study three-player simple games. Let $N = \{1, 2, 3\}$ be a set of players. For any $\pi \in \Pi$ such that $|N^\pi| = 2$, by a standard two-player random-proposer model, there exists a unique subgame perfect equilibrium, which is a cutoff strategy equilibrium with $x_i^\pi = \delta p_i^\pi$ and $q_i^\pi(N^\pi) = 1$ for all $i \in N^\pi$. Thus specifying strategies (x, q) in the initial state is enough for stationary subgame perfect equilibria of three-player games.

We characterize a equilibrium payoff vector for three cases depend on the number of veto players. As examples, we investigate three-party weighted majority games. A tuple $[w^*; w_1, w_2, w_3]$ represents a three-party weighted majority game, in which each party has w_1, w_2, and w_3 votes (or voting

[19] This notion is called *subgame efficiency* by Okada (1996).

strengths) and w^* votes are required to win. Suppose that the initial recognition probabilities are proportional to their votes, and hence $p = \left(\frac{w_1}{w_N}, \frac{w_2}{w_N}, \frac{w_3}{w_N}\right)$.

The proofs appear in Appendix B.1.

6.1.1 No veto player

If there is no veto player, then any two-player coalition is a winning coalition, that is, $\mathbf{W}^m = \{\{1,2\},\{1,3\},\{2,3\}\}$. Furthermore, in an equilibrium (x, q), no matter who proposes first, the winning coalition will be formed immediately, and hence $u_N(x, q) = 1$, or equivalently, $x_N = \delta$.

Proposition 8. *Let (N, v, p, δ) be a three-player simple game with no veto player. If (x, q) is an equilibrium, then $x_1 = x_2 = x_3 = \frac{\delta}{3}$.*

Remark. This result does not depend on the initial recognition probability p and the discount factor δ. This equilibrium payoff vector coincides to both the Shapley-Shubik power index and the core-constrained Nash bargaining solution.

Remark. All the players get the same equilibrium payoff if and only if either all the players are veto or none of them is veto. Recall that if all players are veto in simple games, then it is a unanimity game.

Example 4. Consider a three-party weighted majority game [5;4,3,2] and the initial recognition probability is $(\frac{4}{9}, \frac{3}{9}, \frac{2}{9})$. Since any two parties can win and no party is veto, the equilibrium expected payoffs are $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, independently on δ.

6.1.2 Single veto player

Suppose only player 1 is veto and $\mathbf{W}^m = \{\{1,2\},\{1,3\}\}$. .

Proposition 9. *Let (x, q) be an equilibrium of a three-player simple game with $\mathbf{W}^m = \{\{1,2\},\{1,3\}\}$ and $p = (p_1, p_2, p_3)$ as $\delta \to 1$.*

i. (Strong Solidarity.) If $p_1 \geq \frac{1}{2}$, then $q_2(\{2,3\}) = q_3(\{2,3\}) = 1$ and

$$x_1 = \frac{p_1(3 - 2p_1)}{2 - p_1}, \quad \textit{and} \quad x_2 = x_3 = \frac{(1 - p_1)^2}{2 - p_1}.$$

ii. (Weak Solidarity.) If $p_1 < \frac{1}{2}$, then $0 < q_2(\{2,3\}) < 1$ and $0 < q_3(\{2,3\}) < 1$, and

$$x_1 = \frac{1 + 2p_1}{3}, \quad \textit{and} \quad x_2 = x_3 = \frac{1 - p_1}{3}.$$

Remark. This limiting equilibrium payoff vector depends only on the veto player's recognition probability and the two non-veto players payoffs are the same no matter what their recognition probabilities are. Note that the Shapley-Shubik power index for this game is $(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$. Thus, the equilibrium payoff vector Lorenz-dominates the Shapley-Shubik power index if and only if $p_1 \leq \frac{1}{2}$.

	This Model		Gul	Okada
Buyout Options	O	O	O	X
Strategic Formation	O	O	X	O
Transferable Recog. Prob.	O	X	X	N/A
$\mathbf{W}^m = \{\{1,2\},\{1,3\},\{2,3\}\}$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$
$\mathbf{W}^m = \{\{1,2\},\{1,3\}\}$	$(\frac{5}{9},\frac{2}{9},\frac{2}{9})$	$(\frac{2}{3},\frac{1}{6},\frac{1}{6})$	$(\frac{2}{3},\frac{1}{6},\frac{1}{6})$	$(1,0,0)$
$\mathbf{W}^m = \{\{1,2\}\}$	$(\frac{4}{9},\frac{4}{9},\frac{1}{9})$	$(\frac{1}{2},\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2},0)$
$\mathbf{W}^m = \{N\}$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(\frac{1}{2},\frac{1}{4},\frac{1}{4})^*$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$
$\mathbf{W}^m = \{\{1,2,3\},\{1,3,4\}\}$	$(\frac{1}{3},\frac{1}{6},\frac{1}{3},\frac{1}{6})$	$(\frac{7}{18},\frac{2}{18},\frac{7}{18},\frac{2}{18})$	$(\frac{5}{12},\frac{1}{12},\frac{5}{12},\frac{1}{12})$	$(\frac{1}{2},0,\frac{1}{2},0)$

Table 1: The limiting equilibrium payoff vector for simple games
* Any permutation of $(\frac{1}{2},\frac{1}{4},\frac{1}{4})$ can be an equilibrium payoff vector.

Example 5 (Weak Solidarity)**.** Consider a three-party weighted majority game [6;4,3,2] with $p = \left(\frac{4}{9},\frac{3}{9},\frac{2}{9}\right)$. Since the veto party's weight is less than $\frac{1}{2}$, the smaller two parties form a union each other with positive probability but less than 1 when they are supposed to propose. The limiting equilibrium payoff vector is $\left(\frac{17}{27},\frac{5}{27},\frac{5}{27}\right)$.

Example 6 (Strong Solidarity)**.** Consider a three-party weighted majority game [4;3,2,1] with $p = \left(\frac{1}{2},\frac{1}{3},\frac{1}{6}\right)$. Since the veto party's weight is $\frac{1}{2}$, the smaller two parties always form a union each other when they are supposed to propose. The limiting equilibrium payoff vector is $\left(\frac{2}{3},\frac{1}{6},\frac{1}{6}\right)$.

6.1.3 Two veto players

Suppose that player 1 and player 2 are veto and player 3 is dummy, that is, $\mathbf{W}^m = \{\{1,2\}\}$.

Proposition 10. *Let (x,q) be an equilibrium of a three-player simple game with $\mathbf{W}^m = \{\{1,2\}\}$ and $p = (p_1,p_2,p_3)$ as $\delta \to 1$. Then $q_1(\{1,3\}) > 0$ and $q_2(\{2,3\}) > 0$, and*

$$x_1 = p_1 + \frac{p_3}{3}, \qquad x_2 = p_2 + \frac{p_3}{3}, \quad \text{and} \qquad x_3 = \frac{p_3}{3}.$$

Remark. Even a dummy player can expect a strictly positive payoff, since his chance to be a proposer is valuable.

Example 7. Consider a three-party weighted majority game [5;3,2,1] with $p = \left(\frac{1}{2},\frac{1}{3},\frac{1}{6}\right)$, that is, the two large parties are veto and the smallest party is dummy. The limiting equilibrium payoffs are $\left(\frac{10}{18},\frac{7}{18},\frac{1}{18}\right)$.

6.2 Inequality Comparison

In this subsection, we compare the limiting equilibrium payoff vector with other models. Interestingly, allowing buyout options and strategic coalition formation can reduce inequality of payoff distribution for three-player simple games. As in Gul (1989) and Okada (1996), we assume that the initial recognition probability is uniform, that is, all the players can be a proposer equally likely.

Note that if a bilateral meeting is randomly selected as in Gul (1989), the the limiting equilibrium payoff vector coincides to the Shapley-Shubik power index except for the case of $\mathbf{W}^m = \{N\}$, that is all players are veto. On the other hand, as in Okada (1996), if players have no buyout options, then it coincides to the core-constrained Nash bargaining solution. Due to Proposition 8, 9, and 10, we have the limiting equilibrium payoff vectors with transferable recognition probabilities. We can also calculate the payoff vector when players cannot trade their recognition probabilities. In fact, for three-player simple games with non-transferable recognition probabilities, the limiting equilibrium payoff vector coincides to the Shapley-Shubik power index.[20] The results are summarized in Table 1.

If there is no veto player, that is, $\mathbf{W}^m = \{\{1,2\},\{1,3\},\{2,3\}\}$, all the models expect an egalitarian allocation, in which each player gets $\frac{1}{3}$. If all players are veto, then the payoff vector in the random-bilateral-meeting model is Lorenz-dominated by the others. That is, depriving players of strategic coalition formation yields not only inefficiency but also inequality.[21]

For the other cases, by allowing strategic formation and buyout options (with transferable recognition probabilities), the equilibrium payoff vector Lorenz-dominates the others. That is, if players can freely trade their all the resources and rights including the chances to be a proposer and they can strategically choose their partners to bargain, then the outcome alleviates the inequality relative to both the Shapley-Shubik power index and the core-constrained Nash bargaining solution.

At least for three-player simple games, when players cannot trade their recognition probabilities, buyout options do not reduce the inequality. However, for a 4-player game, buyout option may reduce inequality even with non-transferable recognition probabilities.

Example 8 (4-player Simple Game with Transferable Recognition Probabilities)**.** Consider a 4-player simple game with $N = \{1,2,3,4\}$ and $\mathbf{W} = \{\{1,2,3\},\{1,3,4\},N\}$. That is, odd players are veto and they need at least one even player to win. In a symmetric equilibrium, odd players form a minimal winning coalition, either $\{1,3,4\}$ or $\{1,2,3\}$ with probability $\frac{1}{2}$ for each. On the other hand, even players always form $\{2,4\}$ each other. The limiting equilibrium payoff is $\frac{1}{3}$ for odd players and $\frac{1}{6}$ for even players.

Example 9 (4-player Simple Game with Non-transferable Recognition Probabilities)**.** Consider the same 4-player simple game. Their equilibrium coalition formation strategies are the same as in transferable recognition probabilities case. However, even players' payoffs are worse off. For instance, if player 2 buys out player 4 (but not takes player 4's recognition probability), then player 2 can be a proposer with probability $\frac{1}{3}$ (instead of $\frac{1}{2}$) in the subsequent three-player game. As a result, the limiting equilibrium payoff is $\frac{7}{18}$ for odd players and $\frac{2}{18}$ for even players.

Recall that the Shapley-Shubik power index for this game is $\frac{5}{12}$ for an odd player and $\frac{1}{12}$ for an even player. Note that this power index is the equilibrium payoff vector of Gul (1989)'s model. Thus,

[20] For zero-normalized three-player characteristic function form games with $N = \{1,2,3\}$, if $v(\{1,2\}) + v(\{2,3\}) + v(\{1,3\}) \geq v(N)$, then the limiting equilibrium payoff vector converges to the Shapley value. However, this is not true in general for four-or-more-player games. See Section 8.1 for discussion.

[21] See the example in Gul (1989) p.86 for detail. However, this inefficiency and inequality disappear when player can trade their recognition probabilities.

allowing strategic coalition formation instead of random bilateral meeting as in Gul (1989), the weak players get more payoffs. We conjecture this inequality reduction by buyout options and strategic formation is generically true for simple games. We leave this issue as a future research question.

7 Application II : Wage Bargaining and Labor Union

We consider a three-player game, which is called an employer-employee game. Let $N = \{1, 2, 3\}$ and $v : (2^N \setminus \{\emptyset\}) \to \mathbb{R}$:

$$v(S) = \begin{cases} 1 & \text{if } S = N \\ a & \text{if } S = \{1,2\}, \{1,3\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 \leq a \leq 1$. Player 1 represents an employer or a firm; and player 2 and player 3 represent employees or workers. a refers the first worker's product and $1-a$ reflects the second worker's marginal product. For extreme cases, if $a = 0$, then the game is unanimous; and if $a = 1$, then the game is a simple game with a single veto player.

In the first subsection, we study efficiency and inequality as varying the common discount factor $0 < \delta < 1$, with fixing $a = 1$ and the uniform recognition probability $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. When $\delta > \frac{6}{7}$, the workers form a union with positive probability, which causes inefficiency but reduces inequality. In the second subsection, we investigate the effect of the recognition probability, for fixed $a = 1$ and $\delta \to 1$. The less likely workers are recognized, the more likely they form a union. In the last subsection, we compare our equilibrium payoff vector to other cooperative solution concept, such as the Shapley value and the core-constrained Nash bargaining solution as varying $0 \leq a \leq 1$. We observe that allowing strategic coalition formation can reduce inequality.

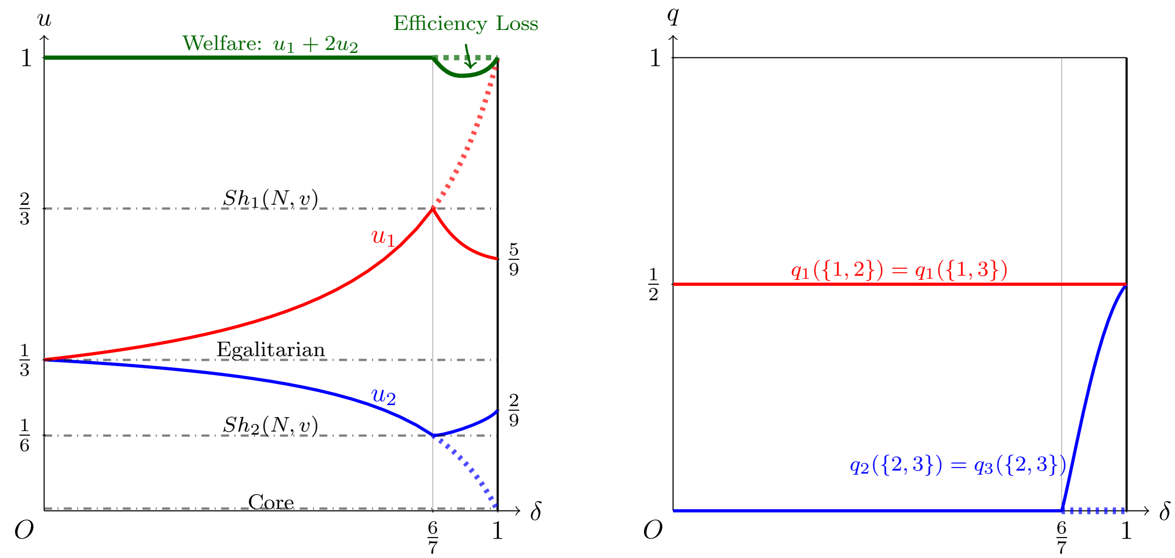
The proofs appear in Appendix B.2.

7.1 Inefficiency and Inequality

If $a = 1$, then there exists a unique core allocation, in which the market clearing wage is zero and the firm takes all the surplus. We assume a uniform recognition probability, that is, each player can be selected as a proposer with probabilities $\frac{1}{3}$. Again we focus on specifying strategies for the initial coalitional state π°. Suppose two workers are identical, we assume symmetric strategies for the workers. An equilibrium (x, q) is symmetric if $x_2 = x_3$, $q_2 = q_3$ and $q_1(\{1,2\}) = q_1(\{1,3\})$. In a symmetric equilibrium (x, q), the excess surplus for each coalition is:

- $e(\{1,2\}, x) = e(\{1,3\}, x) = 1 - (x_1 + x_2)$;
- $e(\{2,3\}, x) = x_2^{(2,\{2,3\})} - 2x_2 = \frac{2}{3}\delta - 2x_2$; and
- $e(N, x) = 1 - (x_1 - 2x_2)$.

Since $x_2 > 0$ in an equilibrium, forming N is strictly dominated by forming either $\{1,2\}$ or $\{1,3\}$ and hence $q_1(\{1,2\}) = q_1(\{1,3\}) = \frac{1}{2}$. Since we assume symmetric strategy for workers, for notational



(a) **Equilibrium Payoffs:** Workers' payoff is greater than their Shapley value; Firm's payoff is less than its Shapley value. If they have no buyout options (dotted lines), then equilibrium payoff vector converges to the unique core allocation.

(b) **Coalition Formation Strategies in an Equilibrium:** Workers form a union with a positive probability and the probability is strictly increasing, if $\delta > \frac{6}{7}$. If the players have no buyout options (a dotted line), then workers never form a union and directly chooses a firm to bargain.

Figure 2: A Wage Bargaining Game ($a = 1$ and $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$)

simplicity, let $q_{23} = q_2(\{2,3\}) = q_3(\{2,3\})$ be the probability that a worker makes a proposal to the other worker. If the proposal between workers is accepted, then a union is formed. Note that the union itself produces nothing, but it could increase workers' bargaining power by unifying their negotiation channel to the firm.

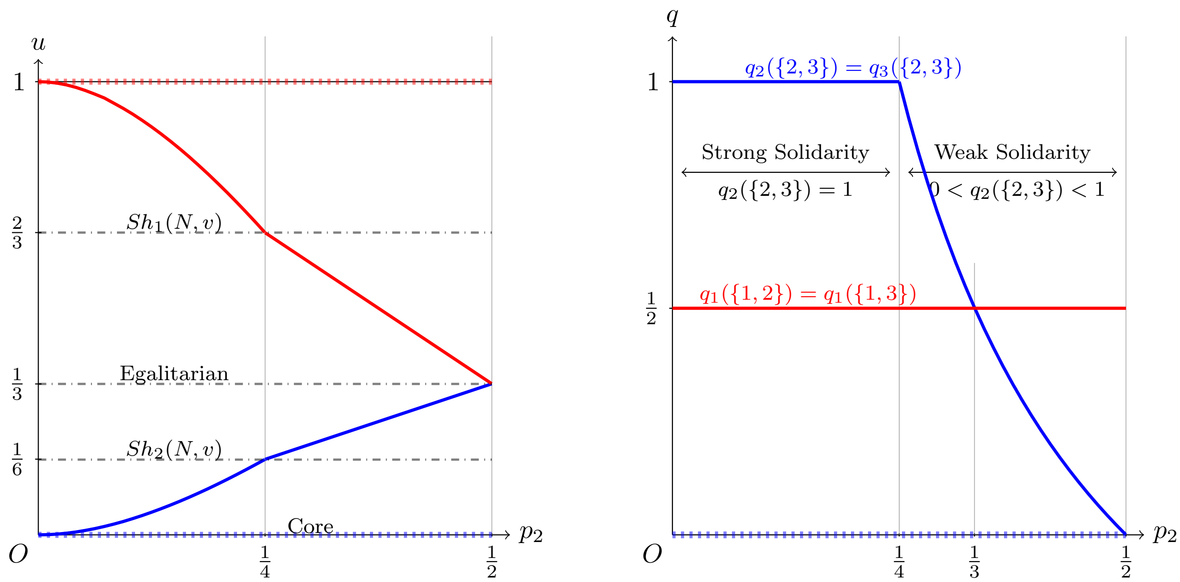
Proposition 11. *There are two types of symmetric equilibria depend on δ.*

i. (No Solidarity.) If $\delta \leq \frac{6}{7}$, then each worker always makes an offer only to the firm and the equilibrium expected payoff is $u_1(\delta) = \frac{2-\delta}{6-5\delta}$ for the firm and $u_2(\delta) = \frac{2-2\delta}{6-5\delta}$ for each worker.

ii. (Weak Solidarity.) If $\delta > \frac{6}{7}$, then each worker makes an offer to each other with strictly positive probability but less than 1. As $\delta \to 1$, q_{23} converges to $\frac{1}{2}$ and the limiting equilibrium payoffs are $\left(\frac{5}{9}, \frac{2}{9}, \frac{2}{9}\right)$.

Note that the unique core allocation is $(1, 0, 0)$ and this allocation. If the players have no buyout option as Okada (2011), then the second type of equilibria (Weak Solidarity) is impossible. Hence, without buyout options, for all $0 < \delta < 1$, the equilibrium expected payoff vector must be $\left(\frac{2-\delta}{6-5\delta}, \frac{2-2\delta}{6-5\delta}, \frac{2-2\delta}{6-5\delta}\right)$ and this converges to the core allocation. However, allowing buyout options to each player, non-core allocations can be obtained as an equilibrium. See Figure 2.

Remark. With non-transferable recognition probabilities, the limiting equilibrium payoff vector is $\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$, which coincides to the Shapley-Shubik power index. See Figure 4 (b).



(a) **Equilibrium Payoffs:** Workers' payoff is strictly positive as long as their recognition probabilities are strictly positive. If they have no buyout options (dotted lines), then workers' equilibrium payoff converges to 0 and the firm takes all the surplus, independently on their recognition probabilities, as long as $\delta \to 1$.

(b) **Equilibrium Payoffs:** The less likely workers are recognized, the more likely they form a union. If $p_2 < \frac{1}{4}$, then workers always form a coalition each other. If they have no buyout options (a dotted line), then workers never form a union and directly chooses a firm to bargain.

Figure 3: A Wage Bargaining Game ($a = 1$, $0 < p_2 = p_3 < \frac{1}{2}$, and $\delta \to 1$)

Remark. When the workers do not make an offer each other, the winning coalition will be formed immediately and hence the sum of equilibrium expected payoffs must be 1, no matter what δ is, as long as $\delta \leq \frac{6}{7}$. However, if $\frac{6}{7} < \delta < 1$, there must be efficiency loss, that is, the sum of equilibrium expected payoffs is strictly less than 1.

7.2 The effect of workers' recognition probability

In this subsection, we assume that the recognition probability is $(1 - 2p, p, p)$, that is each worker can be selected as a proposer with probability $0 < p < \frac{1}{2}$. For ease of exposition, fix $a = 1$ and $\delta = 1$. In a symmetric equilibrium (x, q), the excess surplus for each coalition is:

- $e(\{1,2\}, x) = e(\{1,3\}, x) = 1 - (x_1 + x_2)$;
- $e(\{2,3\}, x) = x_2^{(2,\{2,3\})} - 2x_2 = 2p\delta - 2x_2 = 2p - 2x_2$; and
- $e(N, x) = 1 - (x_1 - 2x_2)$.

Since $\delta = 1$, note that $x_i = u_i$ for each $i \in N$ and $x_N = u_N = 1$. Again, forming N is dominated and hence $q_1(\{1,2\}) = q_1(\{1,3\}) = \frac{1}{2}$. At $\delta = 1$, for any positive workers' recognition probability, each worker form a labor union with positive probability, due to Theorem 1. Now we show that if the workers' recognition probability is lower than a certain level, then they form a union for sure whenever they are supposed to propose.

Proposition 12. *There are two types of equilibria depend on p.*

i. *(Weak Solidarity.) If $\frac{1}{4} < p < \frac{1}{2}$, then each worker makes an offer to each other with probability $q_{23} = q_{32} = \frac{1-2p}{2p}$ and the equilibrium expected payoff is*

$$\left(1 - \frac{4}{3}p,\ \frac{2}{3}p,\ \frac{2}{3}p\right).$$

ii. *(Strong Solidarity.) If $p \leq \frac{1}{4}$, then each worker makes an offer to each other with probability 1 and the equilibrium expected payoff is*

$$\left(1 - \frac{8p^2}{1+2p},\ \frac{4p^2}{1+2p},\ \frac{4p^2}{1+2p}\right).$$

If $p = \frac{1}{4}$, each worker form a union with probability 1 and its equilibrium expected payoff coincides to the Shapley value of the underlying characteristic function form game. As $p \to \frac{1}{2}$, that is the firm has little chance to propose, the equilibrium expected payoff converges to the eqalitarian solution, in which all the players split the surplus equally. As $p \to 0$, that is workers has little chance to propose, the equilibrium payoff vector converges to the core allocation.

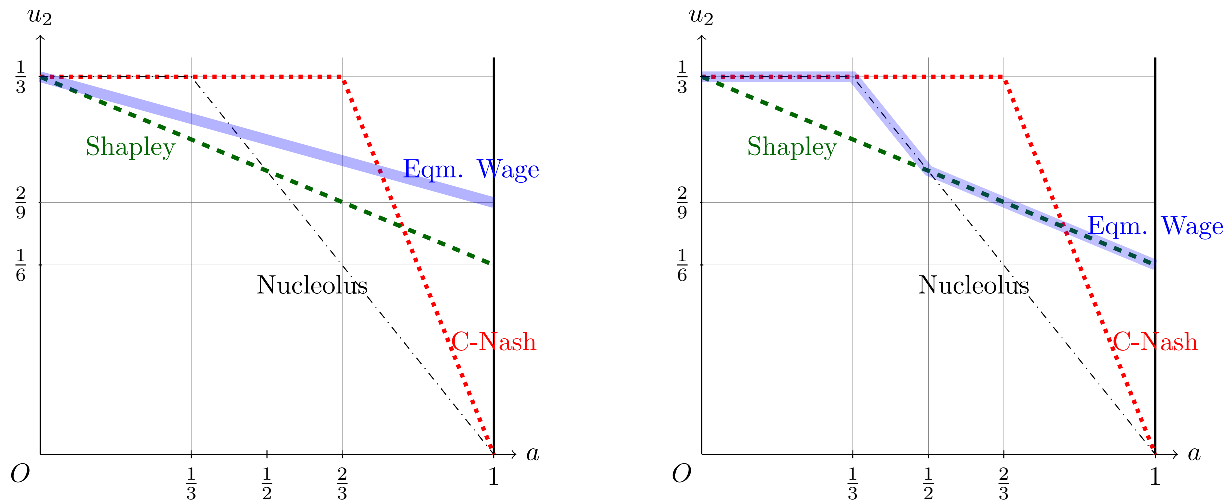
Figure 3 illustrates the effect of buyout option comparing the result with standard models which have no buyout option. In random-proposer models without buyout option, as $\delta \to 1$, the equilibrium payoff must be in the core as long as the core is nonempty. More specifically in Okada (2011), if $\delta \to 1$, workers' payoff is always zero no matter what workers recognition probability. However, as the result of buyout option, workers can form a union and increase their bargaining power by unifying their negotiation channel, and hence they can get a wage more than their marginal product.

7.3 General Cases

In this subsection, we investigate the role of buyout options when a varies between 0 and 1. First, we consider the limiting equilibrium payoff vector with transferable recognition probabilities. As we discussed in Example 2, strict subcoalitions are formed with positive probability in any equilibrium as long as $a > 0$. Thus, the equilibrium payoff vector is generically different from the core-constrained Nash bargaining solution, which can be implemented when a grand-coalition is always immediately formed. See (a) in Figure 4.

Example 10 (Employer-Employee Game with Transferable Recognition Probabilities)**.** The limiting equilibrium payoff is $\frac{3+2a}{9}$ for the firm and $\frac{3-a}{9}$ for each worker. Thus, the equilibrium wage (each worker's payoff) is strictly higher than the Shapley value for any $a > 0$. If $a < \frac{3}{4}$, then the core is relatively large and hence the core-constrained Nash bargaining solution assigns a higher value to each worker than the equilibrium wage in this model. However, if $a > \frac{3}{4}$, that is the core is relatively small, then the equilibrium payoff vector Lorenz-dominates the core-constrained Nash bargaining solution.

However, when players cannot trade their recognition probabilities, the limiting equilibrium payoff may coincide with the core-constrained Nash bargaining solution, nucleolus, or the Shapley value, depend on a. See (b) in Figure 4.



(a) Transferable Recognition Probabilities

(b) Non-transferable Recognition Probabilities

Figure 4: A Wage Bargaining Game: Equilibrium Wage ($p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\delta \to 1$)

Example 11 (Employer-Employee Game with Non-transferable Recognition Probabilities)**.**

i) If $0 \leq a \leq \frac{1}{3}$, then all the players get $\frac{1}{3}$, which coincides with the (core-constrained) Nash bargaining solution, Note that the core is relatively large and all the players always immediately form a grand-coalition. Thus, players do not exercise their buyout options.

ii) If $\frac{1}{3} \leq a \leq \frac{1}{2}$, then the firm's payoff is a and each worker's payoff is $\frac{1-a}{2}$, and this coincide to nucleolus. In this case, the firm is indifferent between immediately forming a grand-coalition and sequentially hiring one by one, while each worker still prefers to forming a grand-coalition.

iii) If $\frac{1}{2} \leq a \leq 1$, then the firm expects $\frac{1+a}{3}$ and each worker expects $\frac{2-a}{6}$. In this case, the excess surpluses of all two-player coalitions are the same and greater than that of a grand-coalition. Hence, the equilibrium payoff coincides with the Shapley value.

8 Concluding Remark

8.1 Marginalism and Egalitarianism

Among many cooperative solution concepts, the Shapley value is one of the best-known solutions. After Shapley (1952), its various axiomatic properties have been studied. Due to Young (1985), it is known that the Shapley value is the unique solution which is anonymous and marginalistic, that is, each player's value depends only on her own marginal contributions to the possible subcoalitions. On the other hand, the notion of egalitarianism is also pervasive in the solution part of cooperative game theory. In addition to the equal division rule, which is the simplest form of egalitarianism, the core-constrained Nash bargaining solution (or egalitarianism under participation constraints), which is introduced by Dutta and Ray (1989), has received much attention.

Both of marginalism and egalitarianism are independently studied in noncooperative approaches. As Gul (1989) showed, if bilateral meetings are randomly selected and each meeting ends in agreement, then the stationary subgame perfect equilibrium payoff vector converges to the Shapley value. Similarly, Chatterjee et al. (1993), if a grand-coalition is always immediately formed, then the equilibrium payoff vector converges to the core-constrained Nash bargaining solution. As Okada (1996) confirmed, the core-constrained Nash bargaining solution is robust with respect to protocols; however, the Shapley value crucially depends on the extensive form game in which every bilateral meetings must occur equally likely.[22]

At least for three-player games, with non-transferable recognition probabilities, our noncooperative bargaining model provides an *unifying framework of marginalism and egalitarianism.* If the sum of two-player coalitions' worths is greater than the worth of the grand-coalition, that is, $v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) \geq v(N)$, then the equilibrium payoff vector converges to the Shapley value. On the other hand, if $\frac{v(N)}{3} \geq v(S)$ for all two-player coalition S, then it converges to the egalitarian value. However, the equilibrium payoff vector does not coincide to the Shapley value in general for four-or-more-player games. If players can strategically choose a coalition to bargain, it is generically hard to implement the Shapley value in a noncooperative bargaining model. If some coalitions are more likely selected than the others by players, then all the marginal contributions are not equally evaluated. Thus, the equilibrium outcome would be a version of marginalistic value can be implemented with endogenously determined weights.

On the other hand, with transferable recognition probabilities, the equilibrium payoff vector of our model generically Lorenz-dominates the Shapley value at least three-player games. The equilibrium outcome has *flavor of both marginalism and egalitarianism.*[23] Since players strategically choose a coalition to bargain, only some subcoalitions are considered to be formed and some players' marginal contributions to certain subcoalitions do not affect to their final outcome. Moreover, when they bargain within the endogenously selected coalitions, they split the surplus of the subcoalition in an egalitarian way, as a discount factor converges to 1. In sum, coalitions are selected with respect to endogenously determined weights and the surpluses of the selected coalitions are distributed according to egalitarianism; but after forming a subcoalition, bargaining continues and only the marginal parts of surpluses are considered in subsequent bargaining games. We conjecture that the equilibrium outcome in our model reduce inequality for general characteristic function form games.

[22] For instance, if they can trade their recognition probabilities, then some meetings occur more frequently. See the partnership game in Gul (1989).

[23] To compromise marginalism and egalitarianism in an axiomatic approach, couples of cooperative solution concepts have been introduced, for instance, *solidarity value* (Nowak and Radzik, 1994) and *sequentially two-levelled egalitarianism* (Lee and Driessen, 2012). For some non-concave games in which the Shapley value is not in the core, those solutions select a core allocations.

8.2 Cooperation Restrictions

Our model effectively explains players' alliance behavior which yields transitional subcoalitions and strategic delay in an equilibrium. In addition to strategic reasons, on the other hand, a transitional coalition could also be formed due to cooperation restrictions for some physical, linguistic, or informational reasons. After Aumann and Dreze (1974), cooperative games with cooperation restrictions have been studied extensively. Our noncooperative model is quite flexible so that cooperation restrictions can be embedded in the following way.

Each $i \in N$ has a *set of feasible coalitions* $\mathcal{S}_i$ such that $S \in \mathcal{S}_i$ implies $i \in S \subseteq N$. Let $\mathcal{S} = \times_{i \in N} \mathcal{S}_i$ be a *cooperation structure*. A triple $(N, v, \mathcal{S})$ is an *underlying characteristic function form game with cooperation restrictions*. In each period, then a proposer $i \in N$ chooses a feasible coalition $S \in \mathcal{S}_i$ to bargain given the cooperation structure. Evolution rule for cooperation structure in the subsequent states can be defined in various way. In this paper, we have assumed all the players can form any coalition that contains himself, that is, $\mathcal{S}_i = \{S \subseteq N \mid i \in S\}$ for all $i \in N$.

One specific cooperation structure is imposing bilateral meetings by defining $\mathcal{S}_i = \{\{i, j\} \mid j \in N\}$. That is, in each period a proposer can form a bilateral meeting as in Gul (1989), but a proposer strategically chooses his or her partner. Gul (1989), Hart and Levy (1999), and Gul (1999) show that when the bilateral meeting is randomly selected an equilibrium outcome is converges to Shapley value as the discount factor closes to 1 under certain conditions on a characteristic function form game. It would be interesting to investigate a condition in which the noncooperative model with strategic decision on choosing the partner implements Shapley value.

A cooperation structure is occasionally represented by a network, and such a situation is modeled as a network-restricted game by Myerson (1977). That is, given a network (or a graph) $g = (N, E)$, where E is a set of *communication link*s in N. Then players can communicate only with their neighbors, that is, $\mathcal{S}_i \equiv \{i \in S \subseteq N \mid \forall j \in S \setminus \{i\} \quad ij \in E\}$. The companion paper, Lee (2013), studies network-restricted games and shows, for a network-restricted unanimity, an efficient equilibrium exists for all discount factors if and only if the underlying network is either complete or circular.

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Appendices

A Proofs for Section 3

A.1 Proposition 1

We prove Proposition 1 in the following three steps.

Step 1: *Let $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be an arbitrary stationary strategy profile. There exists a value profile $\mathbf{x}$ induced by $(\boldsymbol{\alpha}, \boldsymbol{\beta})$.*

Fix $i \in N$ and $(\boldsymbol{\alpha}_{-i}, \boldsymbol{\beta}_{-i})$, that is all the other players except for i play the given stationary strategy. Then the player i's problem is to find i's optimal strategy for a stationary discounted dynamic programming. By the fundamental theorem of stochastic dynamic programming, for every state $\pi \in \Pi$, i has a optimal strategy and it induces a value for i. Furthermore, the optimal strategy of i maximizes the expectation of the sum of the current payoff and the discounted value of next period's state, that is the optimal strategy of i is also stationary. □

Step 2: *Let $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be an SSPE and $\mathbf{x}$ be the induced value profile. For all $\pi \in \Pi$, the partial strategy profile (α^π, β^π) is an initial-state stationary subgame perfect $\mathbf{x}^\pi$-equilibrium.*

Denote $\boldsymbol{\alpha}_{|\pi} \equiv \{\alpha^{\pi'}\}_{\pi' \in \Pi_{|\pi}}$, $\boldsymbol{\beta}_{|\pi} \equiv \{\beta^{\pi'}\}_{\pi' \in \Pi_{|\pi}}$, $\boldsymbol{\alpha}^\pi \equiv \boldsymbol{\alpha}_{|\pi} \cup \{\alpha^\pi\}$, and $\boldsymbol{\beta}^\pi \equiv \boldsymbol{\beta}_{|\pi} \cup \{\beta^\pi\}$. The fixed strategy profile $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ induces a value profile $\mathbf{x}$. For each state π, x^π depend only on $\mathbf{x}_{|\pi}$, $\boldsymbol{\alpha}^\pi$, and $\boldsymbol{\beta}^\pi$. Since $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is an SSPE, the corresponding partial strategy profile (α^π, β^π) satisfies the period optimality condition (2) with respect to $\mathbf{x}^\pi$. Thus, (α^π, β^π) is an initial-state stationary subgame perfect $\mathbf{x}^\pi$-equilibrium. □

Step 3: *If there exist $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and $\mathbf{x}$ such that $\mathbf{x}$ is induced by $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and, for all $\pi \in \Pi$, (α^π, β^π) is an initial-state stationary subgame perfect $\mathbf{x}^\pi$-equilibrium, then $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is an SSPE.*

If a coalitional state π is efficient, then the game ends and each active player $i \in N^\pi$ gets $v([i]_\pi)$ from that period on. If a coalitional state π is inefficient, then it must be $n(\pi) \geq 2$. For $\pi \in \Pi$ with $n(\pi) = 2$, there exists a unique subgame perfect equilibrium of Γ^π, for all $i \in N^\pi$,

$$\begin{aligned} \alpha_i^\pi(h) &= (N^\pi, x^\pi) \qquad \text{for all } h \in H(\pi)\text{; and} \\ \beta_i^\pi(h) &= \begin{cases} 1 & \text{if } y_i(h) \geq x_i^\pi \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $x_i^\pi = (1-\delta)v([i]_\pi) + \delta p_i^\pi v(N)$ for each $i \in N$.

Now we consider an arbitrary inefficient state $\pi \in \Pi$ such that $n(\pi) \geq 3$. Suppose, for all the succeeding states $\pi' \in \Pi_{|\pi}$, $(\boldsymbol{\alpha}^{\pi'}, \boldsymbol{\beta}^{\pi'})$ is an SSPE of the subgame with π' and it induces the value of state $x^{\pi'}$. To show that $(\boldsymbol{\alpha}^\pi, \boldsymbol{\beta}^\pi)$ is an SSPE of the subgame with π, suppose all the active players except for an arbitrary i follow the stationary strategy profile $(\boldsymbol{\alpha}^\pi_{-i}, \boldsymbol{\beta}^\pi_{-i})$. The player i faces a stochastic dynamic programming and hence i has an optimal strategy which maximizes the current return plus the sum of discounted future values, or equivalently, solves the condition (2). Therefore, if (α^π, β^π) is an initial-state stationary subgame perfect $\mathbf{x}^\pi$-equilibrium of the subgame starting with π, then $(\boldsymbol{\alpha}^\pi, \boldsymbol{\beta}^\pi)$ is an SSPE of Γ^π.

Induction argument completes the proof. □

A.2 Proposition 2 and Proposition 3

Lemma 7. *Let $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be an SSPE and $\mathbf{x}$ be the induced value profile. For all $\pi \in \Pi$, $\sum_{j \in N^\pi} x_j^\pi < v(N)$.*

Proof. Fix $\pi \in \Pi$. (α^π, β^π) is an initial-state stationary subgame perfect $\mathbf{x}^\pi$-equilibrium of the subgame starting with π. Thus, for all $j \in N^\pi$, it must be $w_j(h, \alpha^\pi, \beta^\pi, \mathbf{x}^\pi) \geq x_j^\pi$ and hence we have

$$\sum_{j \in N^\pi} w_j(h, \alpha^\pi, \beta^\pi, \mathbf{x}^\pi) \geq \sum_{j \in N^\pi} x_j^\pi.$$

The left-hand side, the sum of expected payoffs of all the active players must be less than or equal to $v(N)$. Suppose, for contradiction, $\sum_{j \in N^\pi} x_j^\pi \geq v(N)$. This yields that $\sum_{j \in N^\pi} w_j(h, \alpha^\pi, \beta^\pi, \mathbf{x}^\pi) = 1$, and hence we have $w_j(h, \alpha^\pi, \beta^\pi, \mathbf{x}^\pi) = x_j^\pi$ for all $j \in N^\pi$, since $w_j(h, \alpha^\pi, \beta^\pi, \mathbf{x}^\pi) \geq x_j^\pi$ for all $j \in N^\pi$. However, this contradicts that π is inefficient. $\square$

Lemma 8. *For all $\pi \in \Pi$, $i \in N$, and $h \in H^i(\pi)$ such that*

i) $\bigtimes_{1 \leq \ell \leq i-1} r_\ell(h) = 1$*; and*

ii) $(\forall \ell \in S(h) \setminus \{\phi(h)\})\ \ell \geq i \implies y_\ell(h) > x_\ell^\pi$,

the current proposal $(S(h), y(h))$ will be implemented for sure in any SSPE.

Proof. Fix $\pi \in \Pi$ and we divide the proof into two cases.
Case 1: $i = n$.

subcase 1-1: Suppose that $n \notin S(h) \setminus \{\phi(h)\}$. It must be $\beta_n^\pi(h) = 1$ no matter what $y_n(h)$. Thus, the outcome at the history of $h' = (h, r_n)$ must be $o(h') = (S(h), y(h))$, that is, $(S(h), y(h))$ will be implemented for sure.

subcase 1-2: Suppose that $n \in S(h) \setminus \{\phi(h)\}$. If $\beta_n^\pi(h) = 1$, then the outcome at the history of $h' = (h, r_n)$ is $o(h') = (S(h), y(h))$, and hence the player n's payoff at the state is $y_n(h)$. If $\beta_n^\pi(h) < 1$, then the outcome at the history of $h' = (h, r_n)$ will be $o(h') = \pi$ with positive probability, and hence they face the same state π in which the player n's value is x_n^π. Thus, for the player n, $\beta_n^\pi(h) = 1$ is optimal at the history h and hence $(S(h), y(h))$ is implemented.

Case 2: $i \leq n - 1$.
As induction hypothesis, suppose that, for all $j > i$ and all $h \in H^j(\pi)$ if $\left[\bigtimes_{1 \leq \ell \leq j-1} r_\ell(h) = 1\right]$ and $\left[(\forall \ell \in S(h) \setminus \{\phi(h)\})\ \ell \geq j \implies y_\ell(h) > x_\ell^\pi\right]$, then $(S(h), y(h))$ is implemented for sure.

subcase 2-1: Suppose that $i \notin S(h) \setminus \{\phi(h)\}$. It must be $\beta_i^\pi(h) = 1$ no matter what $y_i(h)$. Thus, the outcome at the history of $h' = (h, r_i, \cdots, r_n)$ must be $o(h') = (S(h), y(h))$, that is, $(S(h), y(h))$ will be implemented for sure.

subcase 2-2: Suppose that $i \in S(h) \setminus \{\phi(h)\}$. If $\beta_i^\pi(h) = 1$, then the outcome at the history of $h' = (h, r_i, \cdots, r_n)$ is $o(h') = (S(h), y(h))$, and hence the player i's payoff at the state is $y_n(h)$. If $\beta_i^\pi(h) < 1$, then the outcome at the history of $h' = (h, r_i, \cdots, r_n)$ will be $o(h') = \pi$ with positive probability, and hence they face the same state π in which the player i's value is x_i^π. Thus, for the player i, $\beta_i^\pi(h) = 1$ is optimal for the player i at the history h and hence $(S(h), y(h))$ is implemented. $\square$

Lemma 9. *For all $\pi \in \Pi$, $i \in N$, and $h \in H^i(\pi)$ such that*

i) $\bigtimes_{1 \le \ell \le i-1} r_\ell(h) = 1$*; and*

ii) $(\exists \ell \in S(h) \setminus \{\phi(h)\})\ \ell \ge i$ *and* $y_\ell(h) < x^\pi_\ell$*,*

the current proposal $(S(h), y(h))$ will never be implemented in any SSPE.

Proof. Fix $\pi \in \Pi$ and we divide the proof into two cases.

Case 1: $i = n \in S(h) \setminus \{\phi(h)\}$ and $y_n(h) < x^\pi_n$.

If $\beta^\pi_n(h) < 1$, then the outcome at the history of $h' = (h, r_n)$ will be $o(h') = (S(h), y(h))$ with positive probability, and hence the player n's payoff at the state is $y_n(h)$, which less than the stationary value x^π_n. Thus, $\beta^\pi_n(h) = 0$ is optimal for the player n and the current proposal $(S(h), y(h))$ is not implemented.

Case 2: $i \le n-1$. As induction hypothesis, suppose that, for any $h \in H^{i+1}(\pi)$, if $\bigtimes_{1 \le \ell \le i} r_\ell(h) = 1$ and there exists $j \in S(h) \setminus \{\phi(h)\} \cap \{j \ge i+1\}$ such that $y_j(h) < x^\pi_j$, then the proposal $(S(h), y(h))$ will not be implemented.

subcase 2-1: If there exists $j \in S(h) \setminus \{\phi(h)\} \cap \{j \ge i+1\}$ such that $y_j(h) < x^\pi_j$, then by the induction hypothesis, the proposal $(S(h), y(h))$ will not be implemented no matter what $\beta^\pi_i(h)$ is.

subcase 2-2: Suppose that $y_j(h) \ge x^\pi_j$ for all $j \in S(h) \setminus \{\phi(h)\} \cap \{j \ge i+1\}$. It must be $i \in S(h) \setminus \{\phi(h)\}$ and $y_i(h) < x^\pi_i$. For all continuation histories of $(h, r_i = 1)$, $(h, r_i = 1, r_{i+1} = 1)$, $\cdots$, $(h, r_i = 1, \cdots, r_{n-1} = 1)$, if $\beta^\pi_\ell(h, r_i = 1, \cdots, r_{\ell-1} = 1) > 0$ for all $\ell = i+1, i+2, \cdots, n$, then $\beta^\pi_i(h) = 0$ is optimal for the player i and the proposal $(S(h), y(h))$ will not be implemented. If there exists $\ell = i+1, i+2, \cdots, n$, such that $\beta^\pi_\ell(h, r_i = 1, \cdots, r_{\ell-1} = 1) = 0$, again the proposal $(S(h), y(h))$ will not be implemented no matter what $\beta^\pi_i(h)$ is. □

For any $S \in 2^N$, define an allocation $\bar{y}^S \in X$ as $\bar{y}^S_j = x^\pi_j$ for all $j \in S$ and $\bar{y}^S_j = 0$ otherwise.

Lemma 10. *Let $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be an SSPE and $\mathbf{x}$ be the induced value profile. For all $\pi \in \Pi$ and $h \in H(\pi)$, if $\alpha^\pi_{\phi(h)}(S, y) > 0$ then $S \in \mathcal{D}^\pi_{\phi(h)}(\mathbf{x})$ and $y = \bar{y}^S$. Furthermore, every proposal is implemented for sure and the proposal gain of $\phi(h)$ is $m^\pi_{\phi(h)}(\mathbf{x})$.*

Proof. Fix $\pi \in \Pi$ and $h \in H(\pi)$. First, by Lemma 9, the proposal gain of $\phi(h)$ in an SSPE $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is less than or equals to $m^\pi_{\phi(h)}(\mathbf{x})$. Suppose, for contraction, that the proposal gain of $\phi(h)$ in an SSPE $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is strictly less than $m^\pi_{\phi(h)}(\mathbf{x})$ and let $\alpha^\pi_{\phi(h)}$ be the proposal strategy. There must exist (S, y) such that $\alpha^\pi_{\phi(h)}(S, y) > 0$ and $y_j \ge x^\pi_j$ for all $j \in S$ and $y_{j'} > x^\pi_{j'}$ for some $j' \in S$. By Lemma 8, $\phi(h)$ can be strictly better off by slightly decreasing j' share in the proposal, which is a contradiction. Thus, for all player $i \in N^\pi$, the proposal gain of the player i in an SSPE $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ equals to $m^\pi_i(\mathbf{x})$. For the proposal gain $m^\pi_i(\mathbf{x})$ in order to be obtained, the player i must make a proposal $(S, \bar{y}^S)$ for any $S \in \mathcal{P}_i(N^\pi)$, that is, the player i chooses S in $\mathcal{D}^\pi_i(\mathbf{x})$. □

Proof of Proposition 2: Let $\mathbf{x}$ be the value profile induced by an arbitrary SSPE $(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

Case 1: For $\pi \in \Pi$ with $n(\pi) = 2$, $\Gamma^\pi = (N^\pi, v, p^\pi, \delta)$ has a unique subgame perfect Nash equilibrium, in which the active players play cutoff strategies.

Case 2: Consider $\pi \in \Pi$ with $n(\pi) \geq 3$. As induction hypothesis, suppose for all the succeeding states $\pi' \in \Pi_{|\pi}$, there exists a cutoff strategy SSPE $(\mathbf{x}^{\pi'}, \mathbf{q}^{\pi'})$ of $\Gamma^{\pi'} = (N^{\pi'}, v, p^{\pi'}, \delta)$. Now we show that $(\mathbf{x}^\pi, \mathbf{q}^\pi)$ is a cutoff strategy SSPE of Γ^π. By Lemma 10, since the proposed allocation is determined by $\mathbf{x}$, a player i's proposal strategy in π can be represented by $q_i^\pi \in \Delta(\mathcal{D}_i^\pi(\mathbf{x}))$, which is a proposal cutoff strategy. Since this proposal must be implemented for sure by Lemma 10, an active player i's expected payoff when i is not selected as a proposer must equal to the value of current state, that is, $x_i^\pi = (1-\delta)v([i]_\pi) + \delta u_i^\pi(\mathbf{x}^\pi, \mathbf{q}^\pi)$. Since all the active proposer plays a cutoff proposal strategy, for all $i \in N^\pi$ and all $h \in H^{i-1}(\pi)$ such that $i \in S(h) \setminus \{\phi(h)\}$, player i's optimal response strategy is $\beta_i^\pi(h) = 1$ if $y_i(h) \geq x_i^\pi$ and otherwise $\beta_i^\pi(h) = 0$. Thus all the respondents follows the cutoff strategies. □

Proof of Proposition 3:

(only-if part) Suppose $(\mathbf{x}, \mathbf{q})$ is an SSPE. Consider $\pi \in \Pi$ and $i \in N^\pi$. By Lemma 10, the player i's equilibrium proposal strategy must maximizes the proposal gain $x_i^{\pi(i,S)} - \sum_{j \in S} x_j^\pi$, and hence it must be $q_i^\pi \in \Delta(\mathcal{D}_i^\pi(\mathbf{x}))$. When i is supposed to response, i can get at most $(1-\delta)v([i]_\pi) + \delta u_i^\pi(\mathbf{x}, \mathbf{q})$ by rejecting any proposal. Thus in equilibrium, each respondent must indifferent between accepting and rejecting, which requires that $x_i^\pi = (1-\delta)v([i]_\pi) + \delta u_i^\pi(\mathbf{x}, \mathbf{q})$.

(if part) Suppose all the players except i follow the given cutoff strategies $(\mathbf{x}_{-i}, \mathbf{q}_{-i})$. For any π such that $i \in N^\pi$, if $x_i^\pi = (1-\delta)v([i]_\pi) + \delta u_i^\pi(\mathbf{x}, \mathbf{q})$, then it is impossible for i to deviate profitably from the given response strategy. When i is supposed to propose, forming a subcoalition which is not in $\mathcal{D}_i^\pi(\mathbf{x})$ is not optimal for i. On the other hand, a proposer i can propose a grand coalition, in which i's proposal gain is

$$x_i^{\pi(i,N^\pi)} - \sum_{j \in N^\pi} x_j^\pi = v(N) - \sum_{j \in N^\pi} x_j^\pi > v(N) - v(N) = 0,$$

where the first equality is from the fact $\pi(i, N^\pi)$ is efficient; and the inequality is from Lemma 7. Thus, given $\mathbf{x}$, we have $m_i^\pi(\mathbf{x}) > 0$ and the proposer i always has a strictly positive proposal gain as long as the current state is inefficient. That is, making an acceptable proposal is strictly better than a proposal which will be rejected. Therefore, in an SSPE, a proposal i makes a proposal $(S, \bar{y}^S)$ with $S \in \Delta(\mathcal{D}_i^\pi(\mathbf{x}))$, which is the proposal cutoff strategy. □

B Proofs for Applications

B.1 Three-Player Simple Games

Lemma 11. *Let (x, q) be an equilibrium of a three-player simple game. For all $\delta \in (0, 1]$ and all $i \in N$, $x_i > 0$.*

Proof. First, suppose for contradiction $x_1 > 0$ and $x_2 = x_3 = 0$. It must be $\mathcal{D}_2(x) = \{\{2,3\}\}$ and player 2 can form a winning coalition with player 3 without any cost. Hence $u_2(x,q) \geq p_2 > 0$ is strictly positive and hence the expected payoff $x_2 = \delta u_2(x,q) > 0$ is also strictly positive, which yields a contradiction. Now suppose that $x_1 > 0$, $x_2 > 0$, and $x_3 = 0$. Then it must be $u_3(x,q) \geq p_3(1-x_2)$. Since $x_2 < \delta$, we have $u_3(x,q) > 0$, which contradicts that $x_3 = 0$. □

Proposition 8. *Let (N,v,p,δ) be a three-player simple game with no veto player. If (x,q) is an equilibrium, then $x_1 = x_2 = x_3 = \frac{\delta}{3}$.*

Proof. Suppose for contraction that $x_1 > x_2$ and $x_1 > x_3$. It must be that $\mathcal{D}_2(x) = \mathcal{D}_3(x) = \{\{2,3\}\}$ and $N \notin \mathcal{D}_1(x)$. Denote that $q_{12} = q_1(\{1,2\})$ and $q_{13} = q_1(\{1,3\})$. Then we have

$$\begin{aligned} u_1(x,q) &= p_1(1 - q_{12}x_2 - q_{13}x_3), \\ u_2(x,q) &= p_2(1 - x_3) + (p_3 + q_{12}p_1)x_2, \\ u_3(x,q) &= p_3(1 - x_2) + (p_2 + q_{13}p_1)x_3. \end{aligned}$$

Summing up, since $q_{12} + q_{13} = 1$, we have $u_N(x,q) = 1 - p_3x_2 - p_2x_3 < 1$, which is a contraction. □

Lemma 12. *Let (N,v) be a three-player weighted majority game with (p,w^*) and there is a single veto player. Let (x,q) be an equilibrium. If $\delta = 1$, then $x_2 = x_3$.*

Proof. Suppose $x_2 > x_3$. Only three cases are possible.
Case 1: $e(\{23\},x) > e(\{13\},x) > e(\{12\},x)$.
It must be $q_{13} = q_{23} = q_{32} = 1$. Thus, the players expected payoffs are:

$$\begin{aligned} x_1 &= p_1(1 - x_3); \\ x_2 &= p_2(p_2 + p_3 - x_3) + p_3x_2; \\ x_3 &= p_3(p_2 + p_3 - x_2) + (p_1 + p_2)x_3. \end{aligned}$$

The second equation yields $(p_1+p_2)x_2 = p_2(p_2+p_3-x_3)$ and the third equation yields $x_3 = p_2+p_3-x_2$. Combining two conditions, we have $p_1 = 0$, which is a contradiction.
Case 2: $e(\{13\},x) > e(\{23\},x) > e(\{12\},x)$.
It must be $q_{13} = q_{23} = q_{32} = 1$. Thus, player 3's expected payoff is

$$x_3 = p_3(1 - x_1) + (p_1 + p_2)x_3.$$

Rearranging the terms, we have $(1 - p_1 - p_2)x_3 = (1 - x_1)p_3$, which implies $x_1 + x_3 = 1$, or $x_2 = 0$. However, this contradicts to Lemma 11.
Case 3: $e(\{13\},x) > e(\{12\},x) > e(\{23\},x)$.
It must be $q_{13} = q_{21} = q_{31} = 1$ and this implies that a winning coalition must be formed immediately. By Theorem 1, the underlying game must be unanimous, which is a contraction. □

Proposition 9. *Let (x,q) be an equilibrium of a three-player simple game with $\mathbf{W}^m = \{\{1,2\},\{1,3\}\}$ and $p = (p_1,p_2,p_3)$ as $\delta \to 1$.*

i. (Strong Solidarity.) If $p_1 \geq \frac{1}{2}$, then $q_2(\{2,3\}) = q_3(\{2,3\}) = 1$ and

$$x_1 = \frac{p_1(3-2p_1)}{2-p_1}, \quad \text{and} \quad x_2 = x_3 = \frac{(1-p_1)^2}{2-p_1}. \tag{27}$$

ii. (Weak Solidarity.) If $p_1 < \frac{1}{2}$, then $0 < q_2(\{2,3\}) < 1$ and $0 < q_3(\{2,3\}) < 1$, and

$$x_1 = \frac{1+2p_1}{3}, \quad \text{and} \quad x_2 = x_3 = \frac{1-p_1}{3}. \tag{28}$$

Proof. i. (Strong Solidarity.) Suppose $q_{23} = q_{32} = 1$. It must be $e(\{2,3\},x) \geq e(\{1,2\},x)$, or $x_1 - x_2 \geq p_1$. Since $x_2 = x_3$ by Lemma 12, the veto player's expected payoff is:

$$\begin{aligned} x_1 &= p_1(1-x_2) + p_2 x_1^{(2,\{2,3\})} + p_3 x_1^{(3,\{2,3\})} \\ &= p_1\left(1 - \left(\frac{1}{2} - \frac{x_1}{2}\right)\right) + (1-p_1)p_1, \end{aligned}$$

which yields (27). The condition $x_1 - x_2 \geq p_1$ requires that $\frac{p_1(3-2p_1)}{2-p_1} - \frac{1-p_1}{3} \geq p_1$. Solving this inequality, p_1 must satisfy $-2p_1^2 + 3p_1 - 1 \geq 0$, or $\frac{1}{2} \leq p_1 \leq 1$. This completes the proof of the first part.

ii. (Weak Solidarity.) Suppose $0 < q_{23} < 1$ and $0 < q_{32} < 1$. It must be $e(\{2,3\},x) = e(\{1,2\},x) = e(\{1,3\},x)$, or $x_1 - x_2 = p_1 = x_1 - x_3$. Solving these equations with $x_N = 1$, we have (28). In this case, the veto player's expected payoff is:

$$\begin{aligned} x_1 &= p_1(1-x_2) + p_2\left(q_{21}x_1 + q_{23}x_1^{(2,\{2,3\})}\right) + p_3\left(q_{31}x_1 + q_{32}x_1^{(3,\{2,3\})}\right) \\ &= p_1(1-x_2) + rx_1 + (1-r)p_1 - p_1^2, \end{aligned} \tag{29}$$

where $r = p_2 q_{21} + p_3 q_{31} > 0$ is the probability that the veto player is nominated by other players. Plugging (28) into (29), it follows that

$$\frac{1+2p_1}{3} = p_1\left(1 - \frac{1-p_1}{3}\right) + r\frac{1+2p_1}{3} + (1-r)p_1 - p_1^2,$$

which yields $r = 1 - 2p_1$. Since $r > 0$, it must be $r = 1 - 2p_1 > 0$, or $p_1 < \frac{1}{2}$. This completes the proof of the second part. □

Lemma 13. *Let (x,q) be an equilibrium of a three-player simple game with $\mathbf{W}^m = \{\{1,2\}\}$. If $\delta = 1$, then*

$$e(\{1,2\},x) = e(\{1,3\},x) = e(\{2,3\},x).$$

Proof. **Step 1:** Suppose that $e(\{1,2\},x) > e(\{1,3\},x)$. It must be $e(\{2,3\},x) \geq e(\{1,2\},x)$, otherwise Theorem 1 is violated. Thus, player 2 is always nominated by other players, and hence we have $x_2 \geq p_2(1-x_1) + (p_2+p_3)x_2$, or equivalently, $x_1 + x_2 \geq 1$ and $x_3 \leq 0$, which is a contraction.

Step 2: Suppose that $e(\{1,3\},x) > e(\{1,2\},x)$.

- If $e(\{2,3\},x) > e(\{1,2\},x)$, then player 3 is always nominated by other players, and hence $x_3 = p_3(1-x_1) + (1-p_3)x_3$, or $x_1 + x_3 = 1$, which is a contradiction.

- If $e(\{1,2\},x) > e(\{2,3\},x)$, then player 1 is always nominated by other players, and hence $x_1 = p_1(1-x_3) + (1-p_1)x_1$, or $x_1 + x_3 = 1$, which is a contradiction.

- If $e(\{1,2\},x) = e(\{2,3\},x)$, then it must be $1 - x_1 = p_2 + p_3 - x_3$, or $p_1 + x_2 = 1$. Since player 1 and player 3 do not nominate player 2, $x_2 = p_2(1-x_1) + (1-p_2)p_2 = p_2(2 - x_1 - p_2)$. Plugging $p_1 + x_2 = 1$, it follows $x_2 = p_2$. Thus, we have $e(\{1,3\},x) = p_1 + p_3 - x_1 - x_3 = (1-p_2) - (1-x_2) = 0$. However, the assumption $e(\{1,3\},x) > e(\{1,2\},x)$ implies $1 - x_1 - x_2 < 0$, which is a contradiction.

Step 3: Suppose that $e(\{1,2\},x) = e(\{1,3\},x)$. This condition implies that

$$1 - x_2 = p_1 + p_3 - x_3 = (1-p_2) - (1 - x_1 - x_2) = x_1 + x_2 - p_2. \tag{30}$$

If $e(\{1,2\},x) = e(\{1,3\},x) > e(\{2,3\},x)$ then player 1 is always nominated by other players, which is a contradiction again. If $e(\{1,2\},x) = e(\{1,3\},x) < e(\{2,3\},x)$, player 1 is not dominated by other players, and hence $x_1 = p_1(1-x_2) + (1-p_1)p_1$. Thus, with (30) and $x_N = 1$, we have $x_1 = \frac{3(1-p_1)p_1}{2-p_1}$, $x_2 = \frac{1-p_1+p_1^2}{2-p_1}$, and $x_3 = \frac{1-3p+2p^2}{2-p}$. Plugging them into the condition $e(\{1,2\},x) < e(\{2,3\},x)$, that is, $1 - x_1 < p_2 + p_3 - x_3$, it must be

$$1 - \frac{3(1-p_1)p_1}{2-p_1} < 1 - p_1 - \frac{1-3p+2p^2}{2-p},$$

or equivalently, $(2p-1)^2 < 0$, which is a contradiction. Thus, in an equilibrium, it must be $e(\{1,2\},x) = e(\{1,3\},x) = e(\{2,3\},x)$. □

Proposition 10. *Let (x,q) be an equilibrium of a three-player simple game with $\mathbf{W}^m = \{\{1,2\}\}$ and $p = (p_1,p_2,p_3)$ as $\delta \to 1$. Then $q_1(\{1,3\}) > 0$ and $q_2(\{2,3\}) > 0$, and*

$$x_1 = p_1 + \frac{p_3}{3}, \qquad x_2 = p_2 + \frac{p_3}{3}, \quad \textit{and} \qquad x_3 = \frac{p_3}{3}. \tag{31}$$

Proof. By Lemma 13, we have $e(\{1,2\},x) = e(\{1,3\},x) = e(\{2,3\},x)$. The first equation implies that $1 - x_1 - x_2 = p_1 + p_3 - x_1 - x_3$, or $x_2 - x_3 = p_2$. The second equation implies that $p_1 + p_3 - x_1 - x_3 = p_2 + p_3 - x_2 - x_3$, or $2x_2 + x_3 = p_2 + p_3$. Solving two conditions $x_2 - x_3 = p_2$ and $2x_2 + x_3 = p_2 + p_3$ with $x_N = p_N = 1$ yields (31). □

B.2 Wage Bargaining and Labor Union

Lemma 14. *(Possibility of solidarity.) If $\delta > \frac{6}{7}$, each worker forms a union with a strictly positive probability.*

Proof. Suppose, for contradiction, $q_{23} = 0$. It must be $e(12,x) \geq e(23,x)$, that is,

$$1 - \frac{2}{3}\delta \geq x_1 - x_2 = \delta(u_1 - u_2). \tag{32}$$

Since $q_{21} = q_{31} = 1$ and $q_{12} = q_{13} = \frac{1}{2}$, their expected payoffs are:

$$u_1 = \frac{1}{3}(1 - \delta u_2) + \frac{2}{3}\delta u_1; \text{ and} \tag{33}$$

$$u_2 = \frac{1}{3}(1 - \delta u_1) + \frac{1}{3}\frac{1}{2}\delta u_2. \tag{34}$$

Solving (33) and (34), we have $u_1 = \frac{2-\delta}{6-5\delta}$ and $u_2 = \frac{2-2\delta}{6-5\delta}$. Plugging u_1 and u_2 into (32), it follows $7\delta^2 - 27\delta + 18 \geq 0$, which is $\delta \leq \frac{6}{7}$. □

Lemma 15. *(Impossibility of strong solidarity.) For all $0 \leq \delta \leq 1$, each worker forms a union with probability less than 1.*

Proof. Suppose, for contradiction, $q_{23} = 1$, then it must be $e(12, x) \leq e(23, x)$. Since $q_{23} = q_{32} = 1$ and $q_{12} = q_{13} = \frac{1}{2}$, their expected payoffs are:

$$u_1 = \frac{1}{3}(1 - \delta u_2) + \frac{2}{3}\delta\frac{1}{3}; \text{ and} \tag{35}$$

$$u_2 = \frac{1}{3}(\delta\frac{2}{3} - \delta u_2) + \frac{1}{3}\delta u_2 + \frac{1}{3}\frac{1}{2}\delta u_2. \tag{36}$$

which yields $u_1 = \frac{6+3\delta-2\delta^2}{18-3\delta}$ and $u_2 = \frac{4\delta}{18-3\delta}$. With u_1 and u_2, the condition $e(12, x) \leq e(23, x)$, that is, $1 - \frac{2}{3}\delta \leq \delta(u_1 - u_2)$, implies that $\delta^2 + 3\delta - 6 \geq 0$. However, this contradicts to $0 \leq \delta \leq 1$. □

Proposition 11. *There are two types of symmetric equilibria depend on δ.*

i. (No Solidarity.) If $\delta \leq \frac{6}{7}$, then each worker always makes an offer only to the firm and the equilibrium expected payoff is $u_1(\delta) = \frac{2-\delta}{6-5\delta}$ for the firm and $u_2(\delta) = \frac{2-2\delta}{6-5\delta}$ for each worker.

ii. (Weak Solidarity.) If $\delta > \frac{6}{7}$, then each worker makes an offer to each other with strictly positive probability but less than 1. As $\delta \to 1$, q_{23} converges to $\frac{1}{2}$ and the limiting equilibrium payoffs are $(\frac{5}{9}, \frac{2}{9}, \frac{2}{9})$.

Proof. The first part is directly from Lemma 14. For the second part, assume that $\delta > \frac{6}{7}$. By Lemma 14 and Lemma 15, then it must be $0 < q_{23} = q_{32} < 1$ and hence $e(12, x) = e(23, x)$, or equivalently

$$1 - \delta u_1 - \delta u_2 = \frac{2}{3}\delta - 2\delta u_2. \tag{37}$$

With $q_{23} = q_{32}$ and $q_{12} = q_{13} = \frac{1}{2}$, their expected payoffs are:

$$u_1 = \frac{1}{3}(1 - \delta u_2) + \frac{2}{3}\left(q_{23}\delta\frac{1}{3} + (1 - q_{23})\delta u_1\right); \text{ and} \tag{38}$$

$$u_2 = \frac{1}{3}(\delta\frac{2}{3} - \delta u_2) + \frac{1}{3}(q_{23}\delta u_2) + \frac{1}{3}\frac{1}{2}\delta u_2. \tag{39}$$

Simultaneously solving the three equations (37), (38), and (39), the unique solution is:

$$q_{23}(\delta) = \frac{\sqrt{\delta^4 - 28\,\delta^3 + 130\,\delta^2 - 180\,\delta + 81} + \delta^2 - 3}{4\,(\delta - 1)\,\delta}$$

$$u_1(\delta) = \frac{\sqrt{\delta^4 - 28\,\delta^3 + 130\,\delta^2 - 180\,\delta + 81} - 3\,\delta^2 + 16\,\delta - 15}{3\,\left(\sqrt{\delta^4 - 28\,\delta^3 + 130\,\delta^2 - 180\,\delta + 81} - \delta^2 + 8\,\delta - 9\right)}$$

$$u_2(\delta) = \frac{2\,(\delta - 1)\,\left(\sqrt{\delta^4 - 28\,\delta^3 + 130\,\delta^2 - 180\,\delta + 81} - 3\,\delta^2 + 10\,\delta - 9\right)}{3\,\delta\,\left(\sqrt{\delta^4 - 28\,\delta^3 + 130\,\delta^2 - 180\,\delta + 81} - \delta^2 + 2\,\delta - 3\right)}.$$

Given this solution as a function of δ, one can observe that $q_{23}(\delta)$ converges to $\frac{1}{2}$; and $u_1(\delta)$ and $u_2(\delta)$ converges to $\frac{5}{9}$ and $\frac{2}{9}$, repectively. □

Proposition 12. *There are two types of equilibria depend on p.*

i. (Weak Solidarity.) If $\frac{1}{4} < p < \frac{1}{2}$, then each worker makes an offer to each other with probability $q_{23} = q_{32} = \frac{1-2p}{2p}$ and the equilibrium expected payoff is

$$\left(1 - \frac{4}{3}p,\ \frac{2}{3}p,\ \frac{2}{3}p\right).$$

ii. (Strong Solidarity.) If $p \leq \frac{1}{4}$, then each worker makes an offer to each other with probability 1 and the equilibrium expected payoff is

$$\left(1 - \frac{8p^2}{1+2p},\ \frac{4p^2}{1+2p},\ \frac{4p^2}{1+2p}\right).$$

Proof. i. (Weak Solidarity.) Suppose that $0 < q_{23} < 1$. It must be $e(23,x) = e(12,x)$, which implies that $x_2 = \frac{2}{3}p$. Each worker's equilibrium expected payoff is

$$x_2 = p(2p - x_2) + pq_{23}x_2 + (1-2p)q_{12}x_2,$$

which implies $x_2 = \frac{4p^2}{1+4p-2pq_{23}}$. It follows, with the condition $e(23,x) = e(12,x)$, that $x_2 = \frac{4p^2}{1+4p-2pq_{23}} = \frac{2}{3}p$, or equivalently, $q_{23} = \frac{1-2p}{2p}$. Plugging $q = \frac{1-2p}{2p}$, we have $x_2 = \frac{2}{3}p$ and $x_1 = 1 - 2x_2 = 1 - \frac{4}{3}p$. Since $q_{23} = \frac{1-2p}{2p}$ is assumed between 0 and 1, it must be $0 < \frac{1-2p}{2p} < 1$. This condition requires that $\frac{1}{4} < p < \frac{1}{2}$, which completes the proof of the first part.

ii. (Strong Solidarity.) Suppose that $q_{23} = 1$. It must be $e(23,x) \geq e(12,x)$, which implies that $x_2 \leq \frac{2}{3}p$. Each worker's equilibrium expected payoff is

$$x_2 = p(2p - x_2) + px_2 + (1-2p)q_{12}x_2,$$

which implies $x_2 = \frac{4p^2}{1+2p}$ and $x_1 = 1 - 2x_2 = 1 - \frac{8p^2}{1+2p}$. Plugging x_2 into the condition $e(23,x) \geq e(12,x)$, it must be $\frac{4p^2}{1+2p} \leq \frac{2}{3}p$, or equivalently, $p \leq \frac{1}{4}$. This completes the proof of the second part. □